A FRACTIONAL MOSER-TRUDINGER TYPE INEQUALITY IN ONE DIMENSION AND ITS CRITICAL POINTS

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Abstract. We show a sharp fractional Moser-Trudinger type inequality in dimension 1, i.e., for any interval \( I \subseteq \mathbb{R} \) and \( p \in (1, \infty) \) there exists \( \alpha_p > 0 \) such that

\[
\sup_{u \in \tilde{H}^{1/2, p}(I)} \int_I e^{\alpha_p |u|^p |u|^{2-p}} dx = C_p |I|,
\]

and \( \alpha_p \) is optimal in the sense that

\[
\sup_{u \in \tilde{H}^{1/2, p}(I)} \int_I h(u) e^{\alpha_p |u|^p |u|^{2-p}} dx = +\infty,
\]

for any function \( h : [0, \infty) \rightarrow [0, \infty) \) with \( \lim_{t \rightarrow \infty} h(t) = \infty \). Here, \( \tilde{H}^{1/2, p}(I) = \{ u \in L^p(\mathbb{R}) : (-\Delta)^{1/4} u \in L^p(\mathbb{R}), \text{supp}(u) \subseteq I \} \). Restricting ourselves to the case \( p = 2 \), we further consider for \( \lambda > 0 \) the functional

\[
J(u) := \frac{1}{2} \int_{\mathbb{R}} |(-\Delta)^{1/4} u|^2 dx - \lambda \int_I \left( e^{\frac{1}{2}u^2} - 1 \right) dx, \quad u \in \tilde{H}^{1/2, 2}(I),
\]

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455
and prove that it satisfies the Palais-Smale condition at any level \( c \in (-\infty, \pi) \). We use these results to show that the equation

\[
(-\Delta)^{\frac{s}{2}} u = \lambda u e^{u^2} \quad \text{in } I,
\]

has a positive solution in \( \tilde{H}^{\frac{s}{2}}(I) \) if and only if \( \lambda \in (0, \lambda_1(I)) \), where \( \lambda_1(I) \) is the first eigenvalue of \( (-\Delta)^{\frac{s}{2}} \) on \( I \). This extends to the fractional case for some previous results proven by Adimurthi for the Laplacian and the \( p \)-Laplacian operators. Finally, with a technique by Ruf, we show a fractional Moser-Trudinger inequality on \( \mathbb{R} \).

1. Introduction

According to a celebrated result in analysis, if \( \Omega \subset \mathbb{R}^n \) is an open set with finite measure \( |\Omega| \), \( k \) is a positive integer smaller than \( n \), and \( p \in [1, \frac{n}{k}) \), then the Sobolev space \( W_0^{k,p}(\Omega) \) embeds continuously into \( L^{\frac{np}{n-kp}}(\Omega) \). It is also known that in the borderline case \( p = \frac{n}{k} \), one has \( W_0^{k,\frac{n}{k}}(\Omega) \not\subset L^\infty(\Omega) \). On the other hand, as shown by Pohozaev [23], Trudinger [34], Yudovich [36] and others, for the case \( k = 1 \), one has

\[
W_0^{1,n}(\Omega) \subset \left\{ u \in L^1(\Omega) : E_\beta(u) := \int_{\Omega} e^{\beta |u|^{\frac{n}{n-1}}} dx < \infty \right\}, \quad \text{for any } \beta < \infty,
\]

and the functional \( E_\beta \) is continuous on \( W_0^{1,n}(\Omega) \). This embedding was complemented with a sharp inequality by Moser [26], the so-called Moser-Trudinger inequality:

\[
\sup_{u \in W_0^{1,n}(\Omega), \|\nabla u\|_{L^n(\Omega)} \leq 1} \int_{\Omega} e^{\alpha_n |u|^{\frac{n}{n-1}}} dx \leq C|\Omega|, \quad \alpha_n := n\omega_{n-1}^{\frac{1}{n-1}},
\]

where \( \omega_{n-1} \) is the volume of the unit sphere in \( \mathbb{R}^n \), and the constant \( \alpha_n \) is sharp.

Several generalizations and applications of the Moser-Trudinger inequality have appeared in the course of the last decades. In this work, we investigate a fractional version of (1.1), and in particular its sharpness, restricting our attention to dimension 1. Moreover, we will consider a functional associated to it and discuss its critical points.

In order to state the main results of the paper, we first introduce some relevant function spaces. Let \( s \in (0,1) \). We consider the space of functions \( L_s(\mathbb{R}) \) defined by

\[
L_s(\mathbb{R}) = \left\{ u \in L^1_{\text{loc}}(\mathbb{R}) : \int_{\mathbb{R}} \frac{|u(x)|}{1 + |x|^{1+2s}} dx < \infty \right\}.
\]


For a function $u \in L^s(\mathbb{R})$, we define $(-\Delta)^s u$ as a tempered distribution as follows:

$$\langle (-\Delta)^s u, \varphi \rangle := \int_{\mathbb{R}} u(-\Delta)^s \varphi \, dx, \quad \varphi \in \mathcal{S},$$

where $\mathcal{S}$ denotes the Schwartz space of rapidly decreasing smooth functions and for $\varphi \in \mathcal{S}$, we set

$$(-\Delta)^s \varphi := \mathcal{F}^{-1}(|\cdot|^{2s} \hat{\varphi}).$$

Here the Fourier transform is defined by

$$\hat{\varphi}(\xi) \equiv \mathcal{F}\varphi(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} \varphi(x) \, dx.$$

Notice that the convergence of the integral in (1.3) follows from the fact that for $\varphi \in \mathcal{S}$ one has

$$|(-\Delta)^s \varphi(x)| \leq C(1 + |x|^{1+2s})^{-1}.$$

For $s \in (0, 1)$ and $p \in [1, \infty]$, we define the Bessel-potential space

$$H^{s,p}(\mathbb{R}) := \left\{ u \in L^p(\mathbb{R}) : (-\Delta)^{\frac{s}{2}} u \in L^p(\mathbb{R}) \right\},$$

and its subspace

$$\tilde{H}^{s,p}(I) := \left\{ u \in L^p(\mathbb{R}) : u \equiv 0 \text{ in } \mathbb{R} \setminus I, \ (-\Delta)^{\frac{s}{2}} u \in L^p(\mathbb{R}) \right\},$$

where $I \subset \mathbb{R}$ is a bounded interval. Both spaces are endowed with the norm

$$\|u\|_{H^{s,p}(\mathbb{R})} := \|u\|_{L^p(\mathbb{R})} + \|(-\Delta)^{\frac{s}{2}} u\|_{L^p(\mathbb{R})}, \quad \|u\|_{\tilde{H}^{s,p}(I)} := \|u\|_{L^p(\mathbb{R})} + \|(-\Delta)^{\frac{s}{2}} u\|_{L^p(\mathbb{R})}.$$

1.1. **A fractional Moser-Trudinger type inequality.** The first result that we shall prove is a fractional Moser-Trudinger type inequality:

**Theorem 1.1.** For any $p \in (1, +\infty)$ set $p' = \frac{p}{p-1}$ and

$$\alpha_p := \frac{1}{2} \left[ 2 \cos \left( \frac{\pi}{2p} \right) \Gamma \left( \frac{1}{p} \right) \right]^{p'}, \quad \Gamma(z) := \int_0^{+\infty} t^{z-1} e^{-t} \, dt. \quad (1.7)$$

Then for any interval $I \subset \mathbb{R}$ and $\alpha \leq \alpha_p$, we have

$$\sup_{u \in \tilde{H}^{\frac{1}{p'},p}(I), \|(-\Delta)^{\frac{1}{2}} u\|_{L^p(I)} \leq 1} \int_I \left( e^{\alpha |u|^{p'}} - 1 \right) \, dx = C_p |I|, \quad (1.8)$$

and $\alpha = \alpha_p$ is the largest constant for which (1.8) holds. In fact for any function $h : [0, \infty) \to [0, \infty)$ with

$$\lim_{t \to \infty} h(t) = \infty, \quad (1.9)$$
we have
\[
\sup_{u \in H^{\frac{1}{2},p}(I), \|(\Delta)^{\frac{1}{4}} u\|_{L^p(I)} \leq 1} \int_I h(u) \left( e^{\alpha p |u|^p} - 1 \right) dx = \infty. \tag{1.10}
\]

**Remark 1.** Notice that in (1.8), instead of the standard $H^{\frac{1}{2},p}_{\bar{\Omega}}$-norm defined in (1.6), we are using the smaller norm
\[
\|u\| := \|(\Delta)^{\frac{1}{4}} u\|_{L^p(I)}.
\]
This turns out to be equivalent to the full norm $\|u\|_{H^{\frac{1}{2},p}_{\bar{\Omega}}}$ on $\tilde{H}^{\frac{1}{2},p}_{\bar{\Omega}}(I)$. This fact does not appear to be obvious, but one can prove it as follows. By Theorem 7.1 in [16] the operator $T : u \mapsto ((\Delta)^{\frac{1}{4}} u)|_I$ is Fredholm from $\tilde{H}^{\frac{1}{2},p}_{\bar{\Omega}}(I)$ into $L^p(I)$. Moreover $T$ is injective by Lemma 2.2 below. This implies that
\[
\|u\|_{H^{\frac{1}{2},p}_{\bar{\Omega}}} \leq C \|Tu\|_{L^p(I)} = C \|u\|^{*}, \quad \text{for every } u \in \tilde{H}^{\frac{1}{2},p}_{\bar{\Omega}}(I).
\]
Recently, A. Iannizzotto and M. Squassina [17, Cor. 2.4] proved a sub-critical version of (1.8) in Theorem 1.1 in the case $p = 2$, namely
\[
\sup_{u \in \tilde{H}^{\frac{1}{2},2}(I) : \|(\Delta)^{\frac{1}{4}} u\|_{L^2(\Omega)} \leq 1} \int_I e^{\alpha u^2} dx \leq C \alpha |I|, \quad \text{for } \alpha < \pi.
\]

1.2. **Palais-Smale condition and critical points.** In the rest of this paper, we will focus on the case $p = 2$, and denote
\[
H := \tilde{H}^{\frac{1}{2},2}(I), \quad \|u\|_H := \|(\Delta)^{\frac{1}{4}} u\|_{L^2(\Omega)}. \tag{1.11}
\]
By Remark 1 also this norm is equivalent to the full $H^{\frac{1}{2},2}$-norm on $\tilde{H}^{\frac{1}{2},2}(I)$. This also follows from the following Poincaré-type inequality (see e.g. [31, Lemma 6]):
\[
\|u\|_{L^2(\Omega)}^2 \leq \frac{1}{\lambda_1(I)} \|(\Delta)^{\frac{1}{4}} u\|^2_{L^2(\Omega)} \quad \text{for } u \in \tilde{H}^{\frac{1}{2},2}(I), \tag{1.12}
\]
where $\lambda_1 > 0$ is the first eigenvalue of $(-\Delta)^{\frac{1}{2}}$ on $\tilde{H}^{\frac{1}{2},2}(I)$, see Lemma A.8 in the appendix.

We now investigate the existence of critical points of functionals associated to inequality (1.8) in the case $p = 2$. Since we often integrate by parts and
\((-\Delta)^s u\) is not in general supported in \(I\) even if \(u \in C_c^\infty(I)\), it is more natural to consider the slightly weaker inequality
\[
\sup_{u \in H, \|u\|_H^2 \leq 2\pi} \int_I \left( e^{\frac{1}{2}u^2} - 1 \right) \, dx = C|I|, \tag{1.13}
\]
where we use the slightly different norm given in (1.11). The reason for using the constant \(\frac{1}{2}\) instead of \(\alpha_2 = \pi\) in the exponential and having \(\|u\|_H^2 \leq 2\pi\) instead of \(\|u\|_H^2 \leq 1\) is mostly cosmetic, and becomes more apparent when studying the blow-up behavior of critical points of functionals associated to (1.13) (see (1.15) below, and compare to [20] and [24]).

We want to investigate the existence of solutions of the non-local equation
\[
(-\Delta)^{\frac{1}{2}} u = \lambda u e^{\frac{1}{2}u^2} \quad \text{in} \ I, \quad u \equiv 0 \text{ in} \ \mathbb{R} \setminus I, \tag{1.14}
\]
which is the equation satisfied by critical points of the functional \(E : M_\Lambda \to \mathbb{R}\), where
\[
E(u) = \int_I \left( e^{\frac{1}{2}u^2} - 1 \right) \, dx, \quad M_\Lambda := \{u \in H : \|u\|_H^2 = \Lambda\},
\]
\(\Lambda > 0\) is given, \(\lambda\) is a Lagrange multiplier and \(E\) is well defined on \(M_\Lambda\) thanks to Lemma 2.3 below. Since with this variational interpretation of (1.14) it is not possible to prescribe \(\lambda\), we will follow the approach of Adimurthi and see solutions of (1.14) as critical points of the functional
\[
J : H \to \mathbb{R}, \quad J(u) = \frac{1}{2}\|u\|_H^2 - \lambda \int_I \left( e^{\frac{1}{2}u^2} - 1 \right) \, dx. \tag{1.15}
\]
Again \(J\) is well-defined on \(H\) by Lemma 2.3. Moreover it is differentiable by Lemma 2.5 below, and its derivative is given by
\[
\langle J'(u), v \rangle := \frac{d}{dt} J(u + tv) \bigg|_{t=0} = (u, v)_H \lambda \int_I u e^{\frac{1}{2}u^2} \, dx,
\]
for any \(u, v \in H\), where
\[
(u, v)_H := \int_\mathbb{R} (-\Delta)^{\frac{1}{2}} u (-\Delta)^{\frac{1}{2}} v \, dx.
\]
In particular, we have that if \(u \in H\) and \(J'(u) = 0\), then \(u\) is a weak solution of Problem (1.14) in the sense that
\[
(u, v)_H = \lambda \int_I u e^{\frac{1}{2}u^2} \, dx, \quad \text{for all} \ v \in H. \tag{1.16}
\]
That this Hilbert-space definition of (1.14) is equivalent to the definition in sense of tempered distributions given by (1.3) is discussed in the introduction of [20].

To find critical points of $J$, we will follow a method of Nehari, as done by Adimurthi [3]. An important point will be to understand whether $J$ satisfies the Palais-Smale condition or not. We will prove the following:

**Proposition 1.2.** The functional $J$ satisfies the Palais-Smale condition at any level $c \in (-\infty, \pi)$, i.e., any sequence $(u_k)$ with

$$J(u_k) \to c \in (-\infty, \pi), \quad \|J'(u_k)\|_{H'} \to 0 \quad \text{as } k \to \infty,$$

admits a subsequence strongly converging in $H$.

**Theorem 1.3.** Let $I \subset \mathbb{R}$ be a bounded interval and $\lambda_1(I)$ denote the first eigenvalue of $(-\Delta)^{\frac{1}{2}}$ on $H = \dot{H}^{\frac{1}{2}, 2}(I)$. Then for every $\lambda \in (0, \lambda_1(I))$ Problem (1.14) has at least one positive solution $u \in H$ in the sense of (1.16). When $\lambda \geq \lambda_1(I)$ or $\lambda \leq 0$ Problem (1.14) has no non-trivial non-negative solutions.

To prove Theorem 1.3 one constructs a sequence $(u_k)$ which is almost of Palais-Smale type for $J$, in the sense that $J(u_k) \to \bar{c}$ for some $\bar{c} \in \mathbb{R}$ and $\langle J'(u_k), u_k \rangle = 0$. Then a modified version of Proposition 1.2 is used, namely Lemma 3.1 below. In order to do so, it is crucial to show that $\bar{c} < \pi$ (Lemma 4.4 below) and this will follow from (1.10) with $p = 2$ and $h(t) = |t|^2$. Interestingly, in the general case $s > 1$, $n \geq 2$, $p = \frac{n}{s}$, the analog of (1.10) is known only when $s$ is integer or when $h$ satisfies $\lim_{t \to \infty} (t^{-p'}h(t)) = \infty$ (see [25] and Remark 2 below).

Both Proposition 1.2 and Theorem 1.3 were first proven by Adimurthi [3] in dimension $n \geq 2$ with $(-\Delta)^{\frac{1}{2}}$ replaced by the $n$-Laplacian.

Let us briefly discuss the blow-up behavior of solutions to (1.14). Extending previous works in even dimension (see e.g. [4], [13], [24], [28]) the second and third authors and Armin Schikorra [20] studied the blow-up of sequences of solutions to the equation

$$(-\Delta)^{\frac{s}{2}} u = \lambda u^{\frac{2n}{n-s}} \quad \text{in } \Omega \subset \mathbb{R}^n,$$

with suitable Dirichlet-type boundary conditions when $n$ is odd. The moving plane technique for the fractional Laplacian (see [7]) implies that a non-negative solution to (1.14) is symmetric and monotone decreasing from the center of $I$. Then it is not difficult to check that in dimension 1 Theorem 1.5 and Proposition 2.8 of [20] yield:
Theorem 1.4. Fix \(I = (-R,R) \subset \mathbb{R}\) and let \((u_k) \subset H = \vec{H}^{1,2}(I)\) be a sequence of non-negative solutions to
\[
(-\Delta)^{\frac{1}{2}} u_k = \lambda_k u_k e^{\frac{1}{2} u_k^2} \quad \text{in} \ I,
\]
in the sense of (1.16). Let \(m_k := \sup_I u_k\) and assume that
\[
\Lambda := \limsup_{k \to \infty} \|u_k\|_H^2 < \infty.
\]
Then up to extracting a subsequence, we have that either

(i) \(u_k \rightharpoonup u_\infty\) in \(C_\text{loc}^\ell(I) \cap C^0(\bar{I})\) for every \(\ell \geq 0\), where \(u_\infty \in C_\text{loc}^\ell(I) \cap C^0(\bar{I})\) solves
\[
(-\Delta)^{\frac{1}{2}} u_\infty = \lambda_\infty u_\infty e^{\frac{1}{2} u_\infty^2} \quad \text{in} \ I,
\]
for some \(\lambda_\infty \in (0,\lambda_1(I))\), or

(ii) \(u_k \rightharpoonup u_\infty\) weakly in \(H\) and strongly in \(C_\text{loc}^\ell(\bar{I} \setminus \{0\})\) where \(u_\infty\) is a solution to (1.19). Moreover, setting \(r_k\) such that \(\lambda_k r_k m_k^2 e^{\frac{1}{2} m_k^2}\) and
\[
\eta_k(x) := m_k (u_k(r_k x) - m_k) + \log 2, \quad \eta_\infty(x) := \log \left(\frac{2}{1 + |x|^2}\right),
\]
one has \(\eta_k \rightharpoonup \eta_\infty\) in \(C_\text{loc}^\ell(\mathbb{R})\) for every \(\ell \geq 0\) and \(\Lambda \geq \|u_\infty\|_H^2 + 2\pi\).

The function \(\eta_\infty\) appearing in (1.20) solves the equation
\[
(-\Delta)^{\frac{1}{2}} \eta_\infty = e^{\eta_\infty} \quad \text{in} \ \mathbb{R},
\]
which has been recently interpreted in terms of holomorphic immersions of a disk (or the half-plane) by Francesca Da Lio, Tristan Rivi`ere and the third author [11].

Theorem 1.4 should be compared with the two dimensional case, where the analogous equation \(-\Delta u = \lambda u e^{u^2}\) on the unit disk has a more precise blow-up behavior, see e.g. [5], [4], [13], [21].

1.3. A fractional Moser-Trudinger type inequality on the whole \(\mathbb{R}\).

When replacing a bounded interval \(I\) by \(\mathbb{R}\), an estimate of the form (1.8) cannot hold, for instance because of the scaling of (1.8), or simply because the quantity \(\|(-\Delta)^{\frac{1}{2}} u\|_{L^p(\mathbb{R})}\) vanishes on constants. This suggests to use the full Sobolev norm including the term \(\|u\|_{L^p(I)}\) (see Remark 1). This was done by Bernhard Ruf [30] in the case of \(H^{1,2}(\mathbb{R}^2)\). We shall adapt his technique to the case \(H^{1,2}(\mathbb{R})\).
Theorem 1.5. We have
\[
\sup_{u \in H^{1/2,2}(\mathbb{R}), \|u\|_{H^{1/2,2}(\mathbb{R})} \leq 1} \int_{\mathbb{R}} \left( e^{\pi u^2} - 1 \right) \, dx < \infty, \tag{1.21}
\]
where \(\|u\|_{H^{1/2,2}(\mathbb{R})}\) is defined in (1.6). Moreover, for any function \(h : [0, \infty) \rightarrow [0, \infty)\) satisfying
\[
\lim_{t \to \infty} (t^{-2} h(t)) = \infty, \tag{1.22}
\]
we have
\[
\sup_{u \in H^{1/2,2}(\mathbb{R}), \|u\|_{H^{1/2,2}(\mathbb{R})} \leq 1} \int_{\mathbb{R}} h(u) \left( e^{\pi u^2} - 1 \right) \, dx = \infty. \tag{1.23}
\]
In particular, the constant \(\pi\) in (1.21) is sharp.

A main ingredient in the proof of (1.21) is a fractional Pólya-Szegő inequality which seems to be known only in the \(L^2\) setting, being based mainly on Fourier transform techniques.

Open question 1. Does an \(L^p\)-version of Theorem 1.5 hold, i.e., can we replace \(H^{1/2,2}\) with \(H^{1/p, p}\) in (1.21)?

The reason for requiring (1.22) in Theorem 1.5 (contrary to Theorem 1.1, where (1.9) suffices) is that the test functions for (1.23) will be constructed using a cut-off procedure, and due to the non-local nature of the \(H^{1/2,2}\)-norm, giving a precise estimate for the norm of such test functions is difficult.

Open question 2. In analogy with Theorem 1.1, does (1.23) hold for every \(h\) satisfying (1.9)?

In the following sections, we shall prove Theorems 1.1, 1.3 and 1.5, and Proposition 1.2. In the appendix, we collected some useful results about fractional Sobolev spaces and fractional Laplace operators.

2. Theorem 1.1

2.1. Idea of the proof. The following analog of (1.8)
\[
\sup_{u=\delta_p I_1 * f : \text{supp}(f) \subset I, \|f\|_{L^p(I)} \leq 1} \int_I e^{\alpha_p |u|'} \, dx = C_p |I|, \quad I_1(x) := |x|^{\frac{1}{p} - 1}, \tag{2.1}
\]
A fractional Moser-Trudinger type inequality is well-known (also in higher dimension, see e.g. [35, Theorem 3.1]), since it follows easily from the Theorem 2 in [2], up to choosing $c_p$ so that

$$c_p(-\Delta)^{\frac{1}{2p}} I_{\frac{1}{p}} = \delta_0,$$

(2.2)

compare to Lemma 2.1 below.

In (2.1), one requires that the support of $f = (-\Delta)^{\frac{1}{2p}} u$ is bounded; following Adams [2] one would be tempted to write $u = I_{\frac{1}{p}} * (-\Delta)^{\frac{1}{2p}} u$ and apply (2.1), but the support of $(-\Delta)^{\frac{1}{2p}} u$ is in general not bounded, when $u$ is compactly supported.

In order to circumvent this issue, we rely on a Green representation formula of the form

$$u(x) = \int_I G_{\frac{1}{2p}}(x, y)(-\Delta)^{\frac{1}{2p}} u(y) dy,$$

and show that $|G_{\frac{1}{2p}}(x, y)| \leq I_{\frac{1}{p}}(x - y)$ for $x \neq y$. This might follow from the explicit formula of $G_s(x, y)$, which is known on an interval, see e.g. [6] and [10], but we prefer to follow a more self-contained path, only using the maximum principle.

More delicate is the proof of (1.10). We will construct functions $u$ supported in $\bar{I}$ with $(-\Delta)^{\frac{1}{2p}} u = f$ for some prescribed function $f \in L^p(I)$ suitably concentrated. Then with a barrier argument, we will show that $u \in \tilde{H}^{\frac{1}{2p}}(I)$, i.e., $(-\Delta)^{\frac{1}{2p}} u \in L^p(\mathbb{R})$. This is not obvious because $(-\Delta)^{\frac{1}{2p}}$ is a non-local operator and even if $u \equiv 0$ in $I_c$, $(-\Delta)^{\frac{1}{2p}} u$ does not vanish outside $I$, and a priori it could even concentrate on $\partial I$.

Remark 2. An alternative approach to (1.10) uses the Riesz potential and a cut-off function $\psi$, as done in [25] following a suggestion of A. Schikorra. This works in every dimension and for arbitrary powers of $-\Delta$, but it is less efficient in the sense that the $\|(-\Delta)^{s} \psi\|_{L^p}$ is not sufficiently small, and (1.10) (or its higher-order analog) can be proven only for functions $h$ such that $\lim_{t \to \infty} (t^{-p} h(t)) = \infty$. On the other hand, the approach used here to prove (1.10) for every $h$ satisfying (1.9) does not work for higher-order operators, since for instance if for $\Omega \subset \mathbb{R}^4$ we take $u \in W_0^{1,2}(\Omega)$ solving $\Delta u = f \in L^2(\Omega)$, then we do not have in general $u \in W^{2,2}(\mathbb{R}^4)$.

2.2. Proof of Theorem 1.1. By a simple scaling argument it suffices to prove (1.8) for a given interval, say $I = (-1, 1)$.
Lemma 2.1. For \( s \in \left(0, \frac{1}{2}\right) \) the fundamental solution of \((-\Delta)^s\) on \(\mathbb{R}\) is
\[
F_s(x) = \frac{1}{2\cos(s\pi)\Gamma(2s)|x|^{1-2s}},
\]
i.e., \((-\Delta)^s F_s = \delta_0\) in the sense of tempered distributions.

Proof. This follows easily e.g. from Theorem 5.9 in [19].

Lemma 2.2. Fix \( s \in \left(0, \frac{1}{2}\right) \). For any \( x \in I = (-1, 1) \) let \( g_x \in C^\infty(\mathbb{R}) \) be any function with \( g_x(y) = F_s(x - y) \) for \( y \in I^c \). Then there exists \( H_s(x, \cdot) \in \tilde{H}^{s,2}(I) + g_x \) unique solution to
\[
\begin{cases}
(-\Delta)^s H_s(x, \cdot) = 0 & \text{in } I \\
H_s(x, \cdot) = g_x & \text{in } \mathbb{R} \setminus I
\end{cases}
\]
(2.3)
and the function
\[
G_s(x, y) := F_s(x - y) - H_s(x, y), \quad (x, y) \in I \times \mathbb{R}
\]
is the Green function of \((-\Delta)^s\) on \(I\), i.e., for \( x \in I \) it satisfies
\[
\begin{cases}
(-\Delta)^s G_s(x, \cdot) = \delta_x & \text{in } I \\
G(x, y) = 0 & \text{for } y \in \mathbb{R} \setminus I.
\end{cases}
\]
(2.4)
Moreover
\[
0 < G_s(x, y) \leq F_s(x - y) \quad \text{for } y \neq x \in I.
\]
(2.5)
Finally, for any function \( u \in \tilde{H}^{2s,p}(I) \) (\( p \in [1, \infty) \)), we have
\[
u(x) = \int_I G_s(x, y)(-\Delta)^s u(y)dy, \quad \text{for a.e. } x \in I,
\]
(2.6)
where the right-hand side is well defined for a.e. \( x \in I \) thanks to (2.5) and Fubini’s theorem.

Remark 3. The first equations in (2.3) above and in (2.4) below are intended in the sense of distribution, compare to (1.3).

Proof. The existence and non-negativity of \( H_s(x, \cdot) \) for every \( x \in I \) follow from Theorem A.2 and Proposition A.3 in the Appendix. The next claim, namely (2.4), follows at once from Lemma 2.1 and (2.3).

We show now that \( G(x, y) \geq 0 \) for every \((x,y) \in I \times I\). We claim that
\[
\lim_{y \to \pm 1} H_s(x, y) = H_s(x, \pm 1) = F_s(x \mp 1),
\]
(2.7)
hence $G_s(x, y) \to 0$ as $y \to \partial I$, and by Silvestre’s maximum principle, Proposition A.6 below, we also have $G_s(x, \cdot) \geq 0$ for every $x \in I$, hence also (2.5) follows. For the proof of (2.7) notice that

$$\tilde{H}_s(x, \cdot) := H_s(x, \cdot) - g_x \in \tilde{H}^{s, 2}(I),$$

satisfies

$$\begin{cases}
(-\Delta)^s \tilde{H}_s(x, \cdot) = -(-\Delta)^s g_x \quad \text{in } I \\
\tilde{H}_s(x, \cdot) = 0 \quad \text{in } \mathbb{R} \setminus I,
\end{cases}$$

and $((-\Delta)^s g_x)|_I \in L^\infty(I)$ by Proposition A.7 (we are using that $g_x \in C^\infty(\mathbb{R})$), hence Proposition A.4 gives $\tilde{H}_s(x, y) \to 0$ as $y \to \partial I$, and (2.7) follows at once.

To prove (2.6), let us start considering $u \in C_c^\infty(I)$. Then, according to (2.4), we have

$$u(x) = \langle \delta_x, u \rangle = \langle (-\Delta)^s G_s(x, \cdot), u \rangle = \int_I G_s(x, y) (-\Delta)^s u(y) dy.$$ 

Given now $u \in \tilde{H}^{2s, p}(I)$, let $(u_k)_{k \in \mathbb{N}} \subset C_c^\infty(I)$ converge to $u$ in $\tilde{H}^{2s, p}(I)$, i.e.,

$$u_k \to u, \quad (-\Delta)^s u_k \to (-\Delta)^s u \quad \text{in } L^p(\mathbb{R})$$

hence in $L^1(I)$, see Lemma A.5. Then

$$u \xrightarrow{L^1(I)} u_k = \int_I G_s(\cdot, y) (-\Delta)^s u_k(y) dy \xrightarrow{L^1(I)} \int_I G_s(\cdot, y) (-\Delta)^s u(y) dy,$$

the convergence on the right following from (2.5) and Fubini’s theorem:

$$\int_I \left| \int_I G_s(x, y) [(-\Delta)^s u_k(y) - (-\Delta)^s u(y)] dy \right| dx$$

$$\leq \int_I \int_I F_s(x - y) |(-\Delta)^s u_k(y) - (-\Delta)^s u(y)| dx dy$$

$$\leq \sup_{y \in I} \| F_s \|_{L^1(I - y)} \| (-\Delta)^s u_k - (-\Delta)^s u \|_{L^1(I)} \to 0,$$

as $k \to \infty$. Since the convergence in $L^1$ implies the a.e. convergence (up to a subsequence), (2.6) follows.

**Proof of Theorem 1.1.** Set $s = \frac{1}{2p}$. From Lemma 2.2, we get

$$0 \leq (2\alpha_p)^\frac{n-1}{p} G_s(x, y) \leq I_p(x - y) = |x - y|^{\frac{1}{p} - 1},$$
where $G_s$ is the Green’s function of the interval $I$ defined in Lemma 2.2. Choosing $f := |(-\Delta)^{\frac{s}{p}} u|_I$ and using (2.5) and (2.6), we bound

\[(2\alpha_p)^{\frac{p-1}{p}} |u(x)| \leq (2\alpha_p)^{\frac{p-1}{p}} \int I G_s(x, y)f(y)dy \leq I_1 * f(x),\]

and (1.8) follows at once from (2.1).

It remains to show (1.10). The proof is based on the construction of suitable test functions and it is split into steps.

**Step 1. Definition of the test functions.** We fix $\tau \geq 1$ and set

\[f(y) = f_\tau(y) := \frac{1}{2\tau} |y|^{-\frac{1}{p}} \chi_{[-\frac{1}{2}, -\tau] \cup [\tau, \frac{1}{2}]}; \quad r := \frac{e^{-\tau}}{2}.\]  

(2.8)

Notice that

\[\|f\|_{L^p} \geq \frac{2}{(2\tau)^p} \int_p^\frac{1}{2} \frac{dy}{y} = \frac{1}{(2\tau)^{p-1}}.\]

Now, let $u = u_\tau \in \tilde{H}^{s,p}(I)$ solve

\[\begin{cases}
(-\Delta)^s u = f & \text{in } I \\
u \equiv 0 & \text{in } I_c,
\end{cases}\]

in the sense of Theorem A.2 in the appendix.

**Step 2. Proving that $u \in \tilde{H}^{2s,p}(I)$.** According to Proposition A.4, $u$ satisfies

\[|u(x)| \leq C \|f\|_{L^\infty}(1 - |x|)^s \quad \text{for } x \in I.\]

(2.10)

We want to prove that $(-\Delta)^s u \in L^p(\mathbb{R})$. Since by Proposition A.7

\[(-\Delta)^s u(x) = C_s \int_I \frac{-u(y)}{|x-y|^{1+2s}} dy, \quad \text{for } |x| > 1,\]

and $u$ is bounded, we see immediately that

\[|(-\Delta)^s u(x)| \leq \frac{C}{|x|^{1+2s}}, \quad \text{for } |x| \geq 2,\]

hence,

\[\|(-\Delta)^s u\|_{L^q(\mathbb{R}\setminus[-2,2])} < \infty \quad \text{for every } q \in [1, \infty).\]

(2.11)

Now, we claim that

\[\|(-\Delta)^s u\|_{L^q([-2,2]\setminus[-1,1])} < \infty, \quad q = \max\{p, 2\}.\]

(2.12)

Again using Proposition A.7, (2.10) and translating, we have

\[(I) := \|(-\Delta)^s u\|_{L^q([-2,2]\setminus[-1,1])} < \infty, \quad q = \max\{p, 2\}.
\]

(2.12)
A fractional Moser-Trudinger type inequality \[ \leq C \left( \int_{-1}^{0} \left( \int_{0}^{2} \frac{y^s dy}{(y-x)^{1+2s}} \right)^{\frac{1}{q}} \, dx \right)^{\frac{1}{q}}, \]
and using the Minkowski inequality
\[ \left( \int_{A_1} \int_{A_2} |F(x,y)|^q \, dx \right)^{\frac{1}{q}} \leq \int_{A_2} \left( \int_{A_1} |F(x,y)|^q \, dx \right)^{\frac{1}{q}} \, dy, \]
we get
\[ (I) \leq C \int_{0}^{2} y^s \left( \int_{-1}^{0} \frac{dy}{(y-x)^{(1+2s)q}} \right)^{\frac{1}{q}} \, dy \leq C \int_{0}^{2} \frac{dy}{y^{1+s-\frac{1}{q}}} < \infty, \]
since \( 1 + s - \frac{1}{q} < 1 \). This proves (2.12).

To conclude that \((-\Delta)^s u \in L^p(\mathbb{R})\) it remains to show that \((-\Delta)^s u\) does not concentrate on \(\partial I = \{-1,1\}\), in the sense that the distribution defined by
\[ \langle T, \varphi \rangle := \int_{\mathbb{R}} u(-\Delta)^s \varphi dx - \int_{I} f \varphi dx - C_s \int_{I} \int_{\mathbb{R}} \frac{-u(y)}{|x-y|^{1+2s}} \, dy \varphi(x) \, dx \]
\[ =: \langle T_1, \varphi \rangle - \langle T_2, \varphi \rangle - \langle T_3, \varphi \rangle \quad \text{for} \ \varphi \in C_0^\infty(\mathbb{R}), \]
vanishes. Notice that \(\langle T, \varphi \rangle = 0\) for \(\varphi \in C_0^\infty(\mathbb{R} \setminus \partial I)\), since \(T_1 = (-\Delta)^s u\), while
\[ \langle T_2, \varphi \rangle = \langle (-\Delta)^s u, \varphi \rangle, \quad \langle T_3, \varphi \rangle = 0 \quad \text{for} \ \varphi \in C_0^\infty(I), \]
by (2.9), and
\[ \langle T_2, \varphi \rangle = 0, \quad \langle T_3, \varphi \rangle = \langle (-\Delta)^s u, \varphi \rangle \quad \text{for} \ \varphi \in C_0^\infty(I^c), \]
by Proposition A.7, and for \(\varphi \in C_0^\infty(\mathbb{R} \setminus \partial I)\), we can split \(\varphi = \varphi_1 + \varphi_2\) with \(\varphi_1 \in C_0^\infty(I)\) and \(\varphi_2 \in C_0^\infty(I^c)\). In particular, \(\text{supp}(T) \subset \partial I\).

It is easy to see that \(T_1\) is a distribution of order at most 1, i.e.,
\[ \left| \int_{\mathbb{R}} u(-\Delta)^s \varphi dx \right| \leq C \| \varphi \|_{C^1(\mathbb{R})}, \quad \text{for every} \ \varphi \in C_0^\infty(\mathbb{R}), \]
(use for instance Proposition A.7), and that \(T_2\) and \(T_3\) are distributions of order zero, i.e.,
\[ |\langle T_i, \varphi \rangle| \leq C \| \varphi \|_{L^\infty(\mathbb{R})} \quad \text{for} \ i = 2,3. \]
Since \(\text{supp}(T) \subset \partial I\) it follows from Schwartz’s theorem (see e.g. [8, Sec. 6.1.5]) that
\[ T = \alpha \delta_{-1} + \beta \delta_1 + \tilde{\alpha} D \delta_{-1} + \tilde{\beta} D \delta_1, \quad \text{for some} \ \alpha, \beta, \tilde{\alpha}, \tilde{\beta} \in \mathbb{R}, \]
where \(\langle D \delta_{x_0}, \varphi \rangle := -\langle \delta_{x_0}, \varphi' \rangle = -\varphi'(x_0)\) for \(\varphi \in C_0^\infty(\mathbb{R})\).
In order to show that \( \tilde{\alpha} = 0 \), take \( \varphi \in C_c^\infty(\mathbb{R}) \) with
\[
supp(\varphi) \subset (-1, 1), \quad \varphi'(0) = 1, \quad \varphi(0) = 0,
\]
and rescale it by setting for \( \varphi_\lambda(-1 + x) = \lambda \varphi(\lambda^{-1} x) \) for \( \lambda > 0 \). Since \( T_2 \) and \( T_3 \) have order 0 it follows
\[
|\langle T_i, \varphi_\lambda \rangle| \leq C \lambda \to 0 \quad \text{as} \quad \lambda \to 0, \quad \text{for} \quad i = 2, 3.
\]
As for \( T_1 \), using Proposition A.7, we get
\[
\frac{\langle T_1, \varphi_\lambda \rangle}{C_s} = \int_{(B_{2\lambda}(-1))c} u(x) \int_{B_{\lambda}(-1)} \frac{-\varphi_\lambda(y)}{|x - y|^{1 + 2s}} dy dx
\]
\[
+ \int_{B_{2\lambda}(-1)} u(x) \int_{(B_{4\lambda}(-1))c} \frac{\varphi_\lambda(x)}{|x - y|^{1 + 2s}} dy dx
\]
\[
+ \int_{B_{2\lambda}(-1)} u(x) \int_{B_{4\lambda}(-1)} \frac{\varphi_\lambda(x) - \varphi_\lambda(y)}{|x - y|^{1 + 2s}} dy dx
\]
\[
=: (I) + (II) + (III).
\]
Since \( \|\varphi_\lambda\|_{L^\infty(\mathbb{R})} = C \varphi \lambda \) and \( u \in L^\infty(\mathbb{R}) \), one easily bounds \( |(I)| + |(II)| \to 0 \) as \( \lambda \to 0 \), and using that \( \sup_{\mathbb{R}} |\varphi_\lambda'| = \sup_{\mathbb{R}} |\varphi'| \), we get
\[
|(III)| \leq \int_{B_{2\lambda}(-1)} |u(x)| \int_{B_{4\lambda}(-1)} \sup_{\mathbb{R}} |\varphi'| \frac{dy dx}{|x - y|^{2s}}
\]
\[
\leq C \lambda^{1 - 2s} \int_{B_{2\lambda}(-1)} |u(x)| dx \to 0 \quad \text{as} \quad \lambda \to 0.
\]
Since for \( \lambda \in (0, 1) \) we have \( \langle T, \varphi \rangle = -\tilde{\alpha} \), by letting \( \lambda \to 0 \), it follows that \( \tilde{\alpha} = 0 \). Similarly one can prove that \( \tilde{\beta} = 0 \).

We now claim that \( \alpha, \beta = 0 \). Consider \( \tilde{u}(x) := u(x) - \alpha F_s(x + 1) - \beta F_s(x - 1) \), and recalling that \( (-\Delta)^s F_s = \delta_0 \), one obtains that
\[
(-\Delta)^s \tilde{u} = T_1 - \alpha \delta - 1 - \beta \delta_1 = T_2 + T_3 \in L^2(\mathbb{R}),
\]
hence with Proposition A.1
\[
\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\tilde{u}(x) - \tilde{u}(y)|^2}{|x - y|^{1 + 2s}} dx dy = [\tilde{u}]_{W^{2s, 2}(\mathbb{R})}^2 = C \|(-\Delta)^s \tilde{u}\|_{L^2(\mathbb{R})}^2 < \infty,
\]
and this gives a contradiction if \( \alpha \neq 0 \) or \( \beta \neq 0 \) since the integral on the left-hand side does not converge in these cases.

Then \( T = 0 \), i.e., \( (-\Delta)^s u =: T_1 = T_2 + T_3 \) and from (2.9), (2.11) and (2.12) we conclude that \( (-\Delta)^s u \in L^p(\mathbb{R}) \), hence \( u \in H^{2s, p}(I) \), as wished.
Step 3: Conclusion. Recalling that \((-\Delta)^s u = f\) in \(I\), from (2.6), we have for \(x \in I\)
\[
u(x) = \int_I G_s(x, y) f(y) dy
\]
(2.13)
\[= \frac{1}{2 \tau (2 \alpha_p)^{\frac{p-1}{p}}} \int_{r < |y| < \frac{1}{2}} \frac{1}{|x - y|^{1 - \frac{1}{p}} |y|^{\frac{1}{p}}} dy - \int_{r < |y| < \frac{1}{2}} H_s(x, y) f(y) dy
\]
\[=: u_1(x) + u_2(x),
\]
where \(H_s(x, y)\) is as in Lemma 2.2.

We now want a lower bound for \(u\) in the interval \([-r, r]\). We fix \(0 < x \leq r\) and estimate
\[
u_1(x) = \frac{1}{2 \tau (2 \alpha_p)^{\frac{p-1}{p}}} \left( \int_r^{\frac{1}{2}} \frac{dy}{y - x} \frac{1}{|y - x|^{1 - \frac{1}{p}} |y|^{\frac{1}{p}}} + \int_r^{-r} \frac{dy}{|y - x|^{1 - \frac{1}{p}} |y|^{\frac{1}{p}}} \right)
\]
\[\geq \frac{1}{2 \tau (2 \alpha_p)^{\frac{p-1}{p}}} \left( \int_r^{\frac{1}{2}} \frac{dy}{y} + \int_r^{\frac{1}{2}} \frac{dy}{y + x} \right)
\]
\[= \frac{1}{2 \tau (2 \alpha_p)^{\frac{p-1}{p}}} \left( 2 \tau + \log \left( \frac{1 + 2x}{1 + x} \right) \right) = \frac{1}{(2 \alpha_p)^{\frac{p-1}{p}}} + O(\tau^{-1}).
\]
Since \(H_s\) is bounded on \([-r, r] \times [-\frac{1}{2}, \frac{1}{2}]\), we have
\[
|u_2(x)| \leq C \int_r^{\frac{1}{2}} f(y) dy \leq C \tau^{-1} \int_0^{\frac{1}{2}} |y|^{-\frac{1}{p}} dy = O(\tau^{-1}), \quad x \in [-r, r].
\]
Then
\[
u = \nu_\tau \geq \frac{1}{(2 \alpha_p)^{\frac{p-1}{p}}} + O(\tau^{-1}) \quad \text{on} \quad [-r, r],
\]
as \(\tau \to \infty\). We now set
\[
w_\tau := (2 \tau)^{\frac{p-1}{p}} \nu_\tau \in \tilde{H}^{1,p} \left( I \right),
\]
so that \(\|(-\Delta)^s w_\tau\|_{L^p(I)} = 1\), we compute
\[
\int_I e^{\alpha_p |w_\tau|^{p'}} dx \geq \int_{-r}^r e^{\tau + O(1)} dx \geq \frac{2 \tau e^\tau}{C} = \frac{1}{C},
\]
and using that \(\inf_{[-r, r]} w_\tau \to \infty\) as \(\tau \to \infty\), we conclude
\[
\lim_{\tau \to \infty} \int_I h(w_\tau) e^{\alpha_p |w_\tau|^{p'}} dx = \infty,
\]
whenever \(h\) satisfies \(\lim_{t \to \infty} h(t) = \infty\). \(\square\)
2.3. A few consequences of Theorem 1.1.

Lemma 2.3. Let \( u \in H \). Then \( u^q e^{pu^2} \in L^1(I) \) for every \( p, q > 0 \).

Proof. Since \( |u|^q \leq C(q) e^{|u|^2} \), it is enough to prove the case \( q = 0 \). Given \( \varepsilon > 0 \) (to be fixed later), by Lemma A.5 there exists \( v \in C_c^\infty(I) \) such that \( \|v - u\|_H^2 < \varepsilon \). Using \( u^2 \leq (v - u)^2 + 2 vu \), we bound

\[
e^{pu^2} \leq e^{p(v-u)^2} e^{2pvu}.
\]

(2.14)

Using the inequality \( |ab| \leq \frac{1}{2} (a^2 + b^2) \), we have

\[
e^{2pvu} \leq e^{\frac{1}{2p} \|u\|_H^2} e^{e^{\frac{v}{\|u\|_H^2}}},
\]

and for \( \varepsilon \) small enough the right-hand side is bounded in \( L^2(I) \) thanks to Theorem 1.1. Still by Theorem 1.1, we have \( e^{p(v-u)^2} \in L^2(I) \) if \( \varepsilon > 0 \) is small enough, hence going back to (2.14) and using that \( v \in L^\infty(I) \) is now fixed, we conclude with Hölder’s inequality that \( e^{pu^2} \in L^1(I) \). \( \Box \)

Lemma 2.4. For any \( q, p \in (1, +\infty) \) the functional

\[
E_{q,p} : H \rightarrow \mathbb{R}, \quad E_{q,p}(u) := \int_I |u|^q e^{pu^2} \, dx,
\]

is continuous.

Proof. Consider a sequence \( u_k \rightarrow u \) in \( H \). By Lemma 2.3 (up to changing the exponents), we have that the sequence \( f_k := |u_k|^q e^{pu_k^2} \) is bounded in \( L^2(I) \). Indeed, it is enough to write \( u_k = (u_k - u) + u \) and use the same estimates as in (2.14) with \( u \) instead of \( v \) and \( u_k \) instead of \( u \). We now claim that \( f_k \rightarrow f \) in \( L^1(I) \). Indeed, up to a subsequence \( u_k \rightarrow u \) a.e., hence, \( f_k \rightarrow f := |u|^q e^{pu^2} \) a.e.

Then considering that since \( f_k \) is bounded in \( L^2(I) \), we have

\[
\int_{\{f_k > L\}} f_k \, dx \leq \frac{1}{L} \int_{\{f_k > L\}} f_k^2 \, dx \leq \frac{C}{L} \rightarrow 0 \quad \text{as } L \rightarrow +\infty,
\]

the claim follows at once from Lemma A.9. \( \Box \)

Lemma 2.5. The functional \( J : H \rightarrow \mathbb{R} \) defined in (1.15) is of class \( C^\infty \).

Proof. This follows easily from Lemma 2.4, since the first term on the right-hand side of (1.15) is simply \( \frac{1}{2} \|u\|_H^2 \), and the derivatives of the second term are continuous thanks to Lemma 2.4. The details, at least to prove that \( J \in C^1(H) \), are essentially as in the proof of Lemma 2.1 of [33]. The higher-order differentials are handled in the same way since they have a similar form, with the non-linear term \( e^{\frac{1}{2} u^2} \) just multiplied by polynomial terms. \( \Box \)
The following lemma is a fractional analog of a well-known result of P-L. Lions [22].

**Lemma 2.6.** Consider a sequence \((u_k) \subset H\) with \(\|u_k\|_H = 1\) and \(u_k \rightharpoonup u\) weakly in \(H\), but not strongly (so that \(\|u\|_H < 1\)). Then if \(u \neq 0\), \(e^{\pi u^2_k}\) is bounded in \(L^p\) for \(1 \leq p < \tilde{p} := (1 - \|u\|^2_H)^{-1}\).

**Proof.** We split
\[
u_k := e^{\pi u^2_k} = v v_{k,1} v_{k,2},\]
where \(v = e^{\pi |u|^2} \in L^p(I)\) for all \(p \geq 1\) by Lemma 2.3, \(v_{k,1} = e^{-2\pi u(u-u_k)}\) and \(v_{k,2} = e^{\pi (u-u_k)^2}\). Notice now that from
\[-2p\pi u(u-u_k) \leq \pi\left(\frac{p^2}{\varepsilon^2} u^2 + \varepsilon^2 (u-u_k)^2\right),\]
we get from Lemma 2.3 and Theorem 1.1 that \(v_{k,1} \in L^q(I)\) for all \(q \geq 1\) if \(\varepsilon > 0\) is small enough (depending on \(q\)). But again from Theorem 1.1 \(v_{2,k}\) is bounded in \(L^p(I)\) for all \(p < \tilde{p}\) since
\[
\|u_k - u\|^2_H = 1 - 2\langle u_k, u \rangle + \|u\|^2_H \to 1 - \|u\|^2_H.
\]
Therefore, by Hölder’s inequality, we have that \(v_k\) is bounded in \(L^p(I)\) for all \(p < \tilde{p}\). \(\square\)

3. Proof of Proposition 1.2

For the proof of Proposition 1.2, we will closely follow [3]. Set
\[
Q(u) := J(u) - \frac{1}{2} \langle J'(u), u \rangle = \lambda \int_I \left(\left(\frac{u^2}{2} - 1\right)e^{\frac{1}{2}u^2} + 1\right) dx. \tag{3.1}
\]

**Remark 4.** Notice that the integrand on the right-hand side of (3.1) is strictly convex and has a minimum at \(u = 0\); in particular
\[
0 = Q(0) < Q(u) \quad \text{for every } u \in H \setminus \{0\}. \tag{3.2}
\]
Furthermore, by Lemma 2.4 the functional \(Q\) is continuous on \(H\) and by convexity \(Q\) is also weakly lower semi-continuous.

Let us also notice that
\[
\lambda \int_I u^2 e^{\frac{1}{2}u^2} dx = \lambda \int_{\{|u| \leq 4\}} u^2 e^{\frac{1}{2}u^2} dx + \lambda \int_{\{|u| > 4\}} u^2 e^{\frac{1}{2}u^2} dx \leq C + \bar{C}Q(u),
\]
where \(C\) and \(\bar{C}\) are positive constants.
and hence, we have
\[ \lambda \int_I u^2 e^{\frac{\lambda}{2} u^2} \, dx \leq C(1 + Q(u)) \text{ for every } u \in H. \quad (3.3) \]

We consider a Palais-Smale sequence \((u_k)_{k \geq 0}\) with \(J(u_k) \to c\). From (1.17), we get
\[ \langle J'(u_k), u_k \rangle = o(1)\|u_k\|_H \text{ as } k \to \infty, \]
and
\[ Q(u_k) = J(u_k) - \frac{1}{2} \langle J'(u_k), u_k \rangle = c + o(1) + o(1)\|u_k\|_H. \quad (3.4) \]

Then with (3.3), we have
\[ \lambda \int_I u_k^2 e^{\frac{\lambda}{2} u_k^2} \, dx \leq C(1 + \|u_k\|_H), \]
hence, using that \(Q(u_k) \geq 0\)
\[ \lambda \int_I \left( e^{\frac{\lambda}{2} u_k^2} - 1 \right) \, dx \leq C(1 + \|u_k\|_H), \]
so that
\[ J(u_k) \geq \frac{1}{2} \|u_k\|_H^2 - C(1 + \|u_k\|_H). \]

This and the boundedness of \((J(u_k))_{k \geq 0}\) yield that the sequence \((u_k)_{k \geq 0}\) is bounded in \(H\), hence, we can extracts a weakly converging subsequence \(u_k \rightharpoonup \tilde{u}\) in \(H\). By the compactness of the embedding \(H \hookrightarrow L^2\) (see e.g. [12, Theorem 7.1], which we can apply thanks to [12, Proposition 3.6], see Proposition A.1), up to extracting a further subsequence, we can assume that \(u_k \to \tilde{u}\) almost everywhere. To complete the proof of the theorem it remains to show that, up to extracting a further subsequence, \(u_k \to \tilde{u}\) strongly in \(H\).

By Remark 4, we have
\[ 0 \leq Q(\tilde{u}) \leq \liminf_{k \to \infty} Q(u_k) = \liminf_{k \to \infty} \left( J(u_k) - \frac{1}{2} \langle J'(u_k), u_k \rangle \right) = c, \quad (3.5) \]
Thus, necessarily, \(c \geq 0\). In other words, the Palais-Smale condition is vacantly true when \(c < 0\) because no sequence can satisfy (1.17).

Clearly (3.5) implies \(Q(u_k) \to Q(\tilde{u}) = 0\). We now claim that
\[ u_k^p e^{\frac{\lambda}{2} u_k^2} \to \tilde{u}^p e^{\frac{\lambda}{2} \tilde{u}^2} \text{ in } L^1(I) \text{ for } 0 \leq p < 2. \quad (3.6) \]
Indeed, up to extracting a further subsequence, from (3.3) and (3.5), we get
\[ \int_{\{|u_k| > L\}} u_k^p e^{\frac{\lambda}{2} u_k^2} \, dx \leq \frac{1}{L^{2-p}} \int_{\{|u_k| > L\}} u_k^2 e^{\frac{\lambda}{2} u_k^2} \, dx = O\left( \frac{1}{L^{2-p}} \right), \]
and (3.6) follows from Lemma A.9 in the appendix.

Let us now consider the case $c = 0$. Since $Q(\tilde{u}) = 0$, hence $\tilde{u} \equiv 0$, with (3.6), we get

$$\lim_{k \to \infty} \|u_k\|_H^2 = 2 \lim_{k \to \infty} \left( J(u_k) + \lambda \int_I \left( e^{\frac{1}{2}u_k^2} - 1 \right) dx \right)$$

(3.7)

$$= 2\lambda \int_I \left( e^{\frac{1}{2}\tilde{u}^2} - 1 \right) dx = 0,$$

so that $u_k \to 0$ is $H$ and the Palais-Smale condition holds in the case $c = 0$ as well.

The last case is when $c \in (0, \pi)$. We will need the following result which is analogue to Lemma 3.3 in [3].

**Lemma 3.1.** Consider a bounded sequence $(u_k) \subset H$ such that $u_k$ converges weakly and almost everywhere to a function $u \in H$. Further assume that:

1. there exists $c \in (0, \pi]$ such that $J(u_k) \to c$;
2. $\|u\|_H^2 \geq \lambda \int_I u^2 e^{\frac{1}{2}u^2} dx$;
3. $\sup_k \int_I u_k^2 e^{\frac{1}{2}u_k^2} dx < \infty$;
4. either $u \not\equiv 0$ or $c < \pi$.

Then

$$\lim_{k \to \infty} \int_I u_k^2 e^{\frac{1}{2}u_k^2} dx = \int_I u^2 e^{\frac{1}{2}u^2} dx.$$

**Proof.** We assume $u_k \to 0$ in $H$ (if $u \equiv 0$ and $c < \pi$ the existence of $\varepsilon > 0$ in (3.8) below is obvious). We then have $Q(u) > 0$. On the other hand, from assumption 2, we get

$$J(u) = \frac{1}{2} \|u\|_H^2 + Q(u) - \frac{\lambda}{2} \int_I u^2 e^{\frac{1}{2}u^2} dx \geq Q(u) > 0.$$

We also know from the weak convergence of $u_k$ to $u$ in $H$, the weakly lower semicontinuity of the norm and (3.6) that

$$J(u) \leq \lim_{k \to \infty} J(u_k) = c,$$

where the inequality is strict, unless $u_k \to u$ strongly in $H$ (in which case the proof is complete). Then one can choose $\varepsilon > 0$ so that

$$\frac{1 + 2\varepsilon}{\pi} < \frac{1}{c - J(u)}.$$

(3.8)
Notice now that if we set $\beta = \lambda \int_I (e^{\frac{1}{2}u^2} - 1) \, dx$, then
$$\lim_{k \to \infty} \|u_k\|_{H^1}^2 = 2c + 2\beta.$$ 
Then multiplying (3.8) by $\frac{1}{2} \|u_k\|_{H^1}^2$, we have for $k$ large enough
$$\frac{1 + \varepsilon}{2\pi} \|u_k\|_{H^1}^2 \leq \tilde{p} := \frac{1 + 2\varepsilon}{2\pi} \lim_{k \to \infty} \|u_k\|_{H^1}^2 < \frac{c + \beta}{c - J(u)} = \left(1 - \frac{\|u\|_{H^1}^2}{2(c + \beta)}\right)^{-1}.$$ 
By Lemma 2.6 applied to $v_k := \frac{u_k}{\|u_k\|_{H^1}}$, we get that the sequence $\exp(\tilde{p} \pi v_k^2)$ is bounded in $L^1(I)$, hence $e^{\frac{1+\varepsilon}{2}u_k^2}$ is bounded in $L^1$. Now, we have that
$$\int_{\{|u_k| > K\}} u_k^2 e^{\frac{1}{2}u_k^2} \, dx = \int_{\{|u_k| > K\}} \left(u_k^2 e^{-\varepsilon u_k^2}\right) e^{\frac{1+\varepsilon}{2}u_k^2} \, dx$$
$$\leq o(1) \int_{\{|u_k| > K\}} e^{\frac{1+\varepsilon}{2}u_k^2} \, dx,$$
with $o(1) \to 0$ as $K \to \infty$, and we conclude with Lemma A.9.

We now claim
$$\|\tilde{u}\|_{H^1}^2 = \lambda \int_I \tilde{u}^2 e^{\frac{1}{2}\tilde{u}^2} \, dx. \tag{3.9}$$
First, we show that $\tilde{u} \not\equiv 0$. So for the sake of contradiction, we assume that $\tilde{u} \equiv 0$. By Lemma 3.1
$$\lim_{k \to \infty} \int_I u_k^2 e^{\frac{1}{2}u_k^2} \, dx = 0.$$ 
Therefore, also using (3.6), we obtain $\lim_{k \to \infty} Q(u_k) = 0$. It follows that
$$0 < c = \lim_{k \to \infty} J(u_k) = \lim_{k \to \infty} \left(Q(u_k) + \frac{1}{2} \langle J'(u_k), u_k \rangle\right) = 0,$$
contradiction, hence, $\tilde{u} \not\equiv 0$.

Fix now $\varphi \in C^0_0(I) \cap H$. We have $\langle J'(u_k), \varphi \rangle \to 0$ as $k \to \infty$, since $(u_k)$ is a Palais-Smale sequence. But, by weak convergence, we have that $(u_k, \varphi)_H \to (\tilde{u}, \varphi)_H$. Now (3.6) implies
$$\int_I \varphi u_k e^{\frac{1}{2}u_k^2} \, dx \to \int_I \varphi \tilde{u} e^{\frac{1}{2}\tilde{u}^2} \, dx, \quad \text{for every } \varphi \in C^0_0(I).$$
Thus, we have
$$(\tilde{u}, \varphi)_H = \lambda \int_I \varphi \tilde{u} e^{\frac{1}{2}\tilde{u}^2} \, dx.$$
By density and the fact that $\tilde{u}e^{\frac{1}{2}u^2} \in L^p$ for all $p \geq 1$, we have that
\[(\tilde{u}, \tilde{u})_H = \lambda \int_I \tilde{u}^2 e^{\frac{1}{2}u^2} dx,\]
hence, (3.9) is proven. Therefore, we are under the assumptions of Lemma 3.1, which yields
\[
\|\tilde{u}\|_H^2 \leq \liminf_{k \to \infty} \|u_k\|_H^2 = 2 \liminf_{k \to \infty} \left[ J(u_k) + \lambda \int_I (e^{\frac{1}{2}u_k^2} - 1) dx \right]
\]
\[
= 2 \liminf_{k \to \infty} \left[ \frac{\lambda}{2} \int_I u_k^2 e^{\frac{1}{2}u_k^2} dx + \frac{1}{2} \langle J'(u_k), u_k \rangle \right] = \lambda \int_I \tilde{u}^2 e^{\frac{1}{2}u^2} dx = \|\tilde{u}\|_H^2.
\]
By Hilbert space theory the convergence of the norms implies that $u_k \to \tilde{u}$ strongly in $H$, and the Palais-Smale condition is proven.

4. PROOF OF THEOREM 1.3

We start by proving the last claim of Theorem 1.3.

Proposition 4.1. Let $u$ be a non-negative non-trivial solution to (1.14) for some $\lambda \in \mathbb{R}$. Then $0 < \lambda < \lambda_1(I)$.

Proof. Let $\varphi_1 \geq 0$ be as in Lemma A.8. Then using $\varphi_1$ as a test function in (1.14) (compare to (1.16)) yields
\[
\lambda_1(I) \int_I u \varphi_1 dx = \lambda \int_I u \varphi_1 e^{\frac{1}{2}|u|^2} dx > \lambda \int_I u \varphi_1 dx.
\]
Hence, $\lambda < \lambda_1$. Using $u$ as test function in (1.14) gives at once $\lambda > 0$. □

The rest of the section is devoted to the proof of the existence part of Theorem 1.3.

Define the Nehari manifold $N(J) := \{u \in H \setminus \{0\}; \langle J'(u), u \rangle = 0\}$. Since, according to (3.1)-(3.2), $J(u) = Q(u) > 0$ for $u \in N(J)$, we have
\[
a(J) := \inf_{u \in N(J)} J(u) \geq 0.
\]

Lemma 4.2. We have $a(J) > 0$.

Proof. Assume that $a(J) = 0$, then there exists a sequence $(u_k) \subset N(J)$ such that $J(u_k) = Q(u_k) \to 0$ as $k \to \infty$. From (3.3), we infer
\[
\sup_{k \geq 0} \int_I u_k^2 e^{\frac{1}{2}u_k^2} dx < \infty, \quad (4.1)
\]
which, again using the fact that \( u_k \in N(J) \), implies that \( \|u_k\|_H \) is bounded. Thus, up to extracting a subsequence, we have that \( u_k \) weakly converges to a function \( u \in H \). From the weak lower semicontinuity of \( Q \), we then get

\[
0 \leq Q(u) \leq \liminf_{k \to \infty} Q(u_k) = 0,
\]

thus, \( J(u) = Q(u) = 0 \) and (3.2) implies \( u \equiv 0 \). On the other hand, we have from (3.7) with \( \tilde{u} \) replaced by \( u \) (which holds with the same proof thanks to (4.1))

\[
\lim_{k \to \infty} \|u_k\|^2_H = 2 \lim_{k \to \infty} \left\{ J(u_k) + \lambda \int_I \left( e^{\frac{1}{2} u_k^2} - 1 \right) dx \right\} = 0,
\]

(4.2)

therefore, we have strong convergence of \( u_k \) to 0.

Now, if we let \( v_k = \frac{u_k}{\|u_k\|_H} \) and up to a subsequence we assume \( v_k \to v \) weakly in \( H \) and almost everywhere, we have

\[
1 = \|v_k\|^2_H = \lim_{k \to \infty} \lambda \int_I e^{\frac{1}{2} v_k^2} v_k^2 dx = \lambda \int_I v^2 dx < \lambda_1 \int_I v^2 dx \leq 1,
\]

(4.3)

where the third equality is justified as follows: From the Sobolev imbedding \( v_k \to v \) in all \( L^p(I) \) for every \( p \in [1, \infty) \), while from (4.2) and Theorem 1.1, we have that for every \( q \in [1, \infty) \) the sequence \( (e^{\frac{1}{2} v_k^2}) \) is bounded in \( L^q(I) \), hence from Hölder’s inequality, we have the desired limit. The last inequality in (4.3) follows from the Poincaré inequality, see (1.12).

Clearly (4.3) is a contradiction, hence \( a(J) > 0 \).

\[\square\]

**Lemma 4.3.** For every \( u \in H \setminus \{0\} \) there exists a unique \( t = t(u) > 0 \) such that \( tu \in N(J) \). Moreover, if

\[
\|u\|^2_H \leq \lambda \int_I u^2 e^{\frac{1}{2} u^2} dx,
\]

(4.4)

then \( t(u) \leq 1 \) and \( t(u) = 1 \) if and only if \( u \in N(J) \).

**Proof.** Fix \( u \in H \setminus \{0\} \) and for \( t \in (0, \infty) \) define the function

\[
f(t) = t^2 \left( \|u\|^2_H - \lambda \int_I u^2 e^{\frac{1}{2} t^2 u^2} dx \right),
\]

which can also be written as

\[
f(t) = t^2 \left( \|u\|^2_H - \lambda \int_I u^2 dx \right) - t^2 \lambda \int_I u^2 e^{\frac{1}{2} t^2 u^2} dx - 1 \int_I dx.
\]

Notice that \( tu \in N(J) \) if and only if \( f(t) = 0 \).
From the inequality $u^2(e^{\frac{1}{2}t^2u^2} - 1) \geq \frac{1}{2}t^2u^4$, we infer
\[
f(t) \leq t^2\left(\|u\|^2_H - \lambda \int_I u^2dx\right) - \frac{1}{2}t^4\lambda \int_I u^4dx,
\]
hence,
\[
\lim_{t \to +\infty} f(t) = -\infty.
\]
Now, notice that the function $t \mapsto (e^{\frac{1}{2}t^2u^2} - 1)$ is monotone increasing on $(0, \infty)$, and by Lemma 2.3, we have $(e^{\frac{1}{2}u^2} - 1) \in L^p(I)$ for all $p \in [1, \infty)$, so that $u^2(e^{\frac{1}{2}u^2} - 1) \in L^1(I)$. Then by the dominated convergence theorem, we get
\[
\lim_{t \to 0} \int_I u^2\left(e^{\frac{1}{2}t^2u^2} - 1\right) dx = 0.
\]
So one has
\[
f(t) = t^2\left(\|u\|^2_H - \lambda \int_I u^2dx\right) + o(t^2) \quad \text{as } t \to 0.
\]
Hence, $f(t) > 0$ for $t$ small, since for $\lambda < \lambda_1(I)$
\[
\|u\|^2_H - \lambda \int_I u^2dx > 0,
\]
(compare the proof of Lemma A.8). Therefore there exists $t = t(u)$ such that $f(t) = 0$, i.e., $tu \in N(J)$. The uniqueness of such $t$ follows noticing that the function
\[
t \mapsto \int_I u^2e^{\frac{1}{2}t^2u^2} dx,
\]
is increasing. Keeping this in mind, if we assume (4.4), then $f(1) \leq 0$, hence $f(t) \leq 0$ for all $t \geq 1$. This implies at once that $t(u) \leq 1$ and $t(u) = 1$ if and only if $u \in N(J)$. □

**Lemma 4.4.** We have $a(J) < \pi$.

**Proof.** Take $w \in H$ such that $\|w\|_H = 1$ and let $t = t(w)$ be given as in Lemma 4.3 so that $tw \in N(J)$. Then
\[
a(J) \leq J(tw) \leq \frac{t^2}{2}\|w\|^2_H = \frac{t^2}{2}.
\]
Now, using the monotonicity of $t \mapsto \int_I w^2e^{\frac{1}{2}t^2w^2} dx$, we have
\[
\lambda \int_I w^2e^{\alpha(J)w^2} dx \leq \lambda \int_I w^2e^{\frac{1}{2}t^2w^2} dx = \frac{t^2\|w\|^2_H}{t^2} = 1.
\]
Thus,
\[
\sup_{\|w\|_{H^1} = 1} \lambda \int_I w^2 e^{a(J)w^2} \, dx \leq 1,
\]
and Theorem 1.1 implies that \( a(J) < \pi \).

**Lemma 4.5.** Let \( u \in N(J) \) be such that \( J'(u) \neq 0 \), then \( J(u) > a(J) \).

**Proof.** We choose \( h \in H \) such that \( \langle J'(u), h \rangle = 1 \), and for \( \alpha \in \mathbb{R} \), we consider the path \( \sigma_t(\alpha) = \alpha u - th, t \in \mathbb{R} \). Remember that by Lemma 2.5 \( J \in C^1(H) \). By the chain rule
\[
\frac{d}{dt} J(\sigma_t(\alpha)) = -\langle J'(\sigma_t(\alpha)), h \rangle,
\]
therefore, if we set \( t = 0, \alpha = 1 \), we find
\[
\frac{d}{dt} J(\sigma_t(\alpha)) \bigg|_{t=0,\alpha=1} = -\langle J'(u), h \rangle = -1.
\]
Hence, there exist \( \delta > 0 \) and \( \varepsilon > 0 \) such that for \( \alpha \in [1 - \varepsilon, 1 + \varepsilon] \) and \( t \in (0, \delta] \)
\[
J(\sigma_t(\alpha)) < J(\sigma_0(\alpha)) = J(\alpha u).
\]
Now, we consider the function \( f \) defined by
\[
f_t(\alpha) = \|\sigma_t(\alpha)\|_{H^1}^2 - \lambda \int_I \sigma_t(\alpha)^2 e^{\frac{1}{2}\sigma_t(\alpha)^2} \, dx,
\]
which is continuous with respect to \( t \) and \( \alpha \) by Lemma 2.4. Notice that since \( u \in N(J) \), we have
\[
f_0(\alpha) = \alpha^2 \int_I u^2 \left( e^{\frac{1}{2}\alpha^2 u^2} - e^{\frac{1}{2}\alpha^2 u^2} \right) \, dx,
\]
and \( f_0(1) = 0 \). Since the function \( \alpha \mapsto u^2 \left( e^{\frac{1}{2}\alpha^2 u^2} - e^{\frac{1}{2}\alpha^2 u^2} \right) \) is decreasing, by continuity we can find \( \varepsilon_1 \in (0, \varepsilon) \) and \( \delta_1 \in (0, \delta) \) such that
\[
f_t(1 - \varepsilon_1) > 0, \quad f_t(1 + \varepsilon_1) < 0 \quad \text{for } t \in [0, \delta_1].
\]
Then if we fix \( t \in (0, \delta_1] \), we can find \( \alpha_t \in [1 - \varepsilon_1, 1 + \varepsilon_1] \) such that \( f_t(\alpha_t) = 0 \), i.e., \( \sigma_t(\alpha_t) \in N(J) \), and from (4.5), we get
\[
a(J) \leq J(\sigma_t(\alpha_t)) < J(\alpha_t u).
\]
Since
\[
\frac{d}{d\alpha} J(\alpha u) = f_0(\alpha),
\]
and \( f_0(\alpha) > 0 \) for \( \alpha < 1 \) and \( f_0(\alpha) < 0 \) for \( \alpha > 1 \), we get \( J(\alpha u) \leq J(u) \) for \( \alpha \in \mathbb{R} \), and we conclude that \( a(J) \leq J(\sigma_t(\alpha_t)) < J(\alpha_t u) \leq J(u) \). \( \Box \)
Proof of Theorem 1.3 (completed). To complete the proof it is enough to show the existence of \( u_0 \in N(J) \) such that \( J(u_0) = a(J) \). We consider then a minimizing sequence \((u_k) \subset N(J)\). We assume that \( u_k \) changes sign. Then since \( u_k \in N(J) \), we have
\[
\|u_k\|_H^2 < \|u_k\|_H^2 = \lambda \int_I u_k^2 e^{\frac{1}{2}u_k^2} dx = \lambda \int_I |u_k|^2 e^{\frac{1}{2}|u_k|^2} dx,
\]
where we used (A.6), hence, by Lemma 4.3 there exists \( t_k = t(|u_k|) < 1 \) such that \( t_k|u_k| \in N(J) \), whence
\[
J(t_k|u_k|) = Q(t_k|u_k|) < Q(|u_k|) = Q(u_k) = J(u_k),
\]
where the inequality in the middle depends on the monotonicity of \( Q \). Hence, up to replacing \( u_k \) with \( t_k|u_k| \), we can assume that the minimizing sequence (still denoted by \((u_k)\)) is made of non-negative functions.

Since \( J(u_k) = Q(u_k) \leq C \), we infer from (3.3)
\[
\int_I u_k^2 e^{\frac{1}{2}u_k^2} dx \leq C,
\]
and for \( u_k \in N(J) \), we get \( \|u_k\|_H \leq C \). Thus, up to a subsequence \( u_k \) weakly converges to a function \( u_0 \in H \), and up to a subsequence the convergence is also almost everywhere.

We claim that \( u_0 \neq 0 \). Indeed if \( u_0 \equiv 0 \), then from (3.6), we have that
\[
(e^{\frac{1}{2}u_k^2} - 1) \rightarrow 0 \quad \text{in} \quad L^1(I).
\]
Thus,
\[
\lim_{k \to \infty} \|u_k\|_H^2 = 2 \lim_{k \to \infty} \left[ J(u_k) + \lambda \int_I \left( e^{\frac{1}{2}u_k^2} - 1 \right) dx \right] = 2a(J).
\]
Then according to Theorem 1.1, since \( a(J) < \pi \), we have that \( e^{\frac{1}{2}u_k^2} \) is bounded in \( L^p \) for some \( p > 1 \), hence weakly converging in \( L^p(I) \) to \( e^{\frac{1}{2}u_0^2} \). From the compactness of the Sobolev embeddings (see [12, Theorem 7.1], which can be applied thanks to Proposition A.1), up to a subsequence \( u_k^2 \rightarrow u_0^2 \) strongly in \( L^p(I) \), hence
\[
\lim_{k \to \infty} \int_I u_k^2 e^{\frac{1}{2}u_k^2} dx = \int_I u_0^2 e^{\frac{1}{2}u_0^2} dx = 0,
\]
and with Lemma 4.2 and (3.1), one gets
\[
0 < a(J) = \lim_{k \to \infty} J(u_k) = \lim_{k \to \infty} Q(u_k) = 0,
\]
which is a contradiction.
Next, we claim that
\[ \|u_0\|_{H^2}^2 \leq \lambda \int_I u_0^2 e^\frac{1}{2}u_0^2 dx. \]
So, we assume by contradiction that this is not the case, i.e.,
\[ \|u_0\|_{H^2}^2 > \lambda \int_I u_0^2 e^\frac{1}{2}u_0^2 dx. \]
Then from Lemma 3.1, Lemma 4.4 and the weak convergence, we have that
\[ \|u_0\|_{H^2}^2 \leq \liminf_{k \to \infty} \|u_k\|_{H^2}^2 = \liminf_{k \to \infty} \lambda \int_I u_k^2 e^\frac{1}{2}u_k^2 dx \]
again leading to a contradiction.

From Lemma 4.3, we have that there exists \( 0 < t \leq 1 \) such that \( tu_0 \in N(J) \).
Taking Remark 4 into account, we get
\[ a(J) \leq J(tu_0) = Q(tu_0) \leq Q(u_0) \leq \liminf_{k \to \infty} Q(u_k) = a(J). \]
It follows that \( t = 1 \), since otherwise the second inequality above would be strict. Then \( u_0 \in N(J) \) and \( J(u_0) = a(J) \). By Lemma 4.5, we have \( J'(u_0) = 0 \). \( \square \)

5. Proof of Theorem 1.5

For \( u \in H^{\frac{1}{2}, 2}(\mathbb{R}) \), we set \(|u|^* : \mathbb{R} \to \mathbb{R}_+\) to be its non-increasing symmetric rearrangement, whose definition we shall now recall. For a measurable set \( A \subset \mathbb{R} \), we define \( A^* = (-|A|/2, |A|/2) \). The set \( A^* \) is symmetric (with respect to 0) and \(|A^*| = |A|\). For a non-negative measurable function \( f \), such that \(|\{x \in \mathbb{R} : f(x) > t\}| < \infty \) for every \( t > 0 \), we define the symmetric non-increasing rearrangement of \( f \) by
\[ f^*(x) = \int_0^\infty \chi_{\{y \in \mathbb{R} : f(y) > t\}}^*(x) dt. \]
Notice that \( f^* \) is even, i.e., \( f^*(x) = f^*(-x) \) and non-increasing (on \([0, \infty)\)).

We will state here the two properties that we shall use in the proof of Theorem 1.5. The following one is proven e.g. in [19, Section 3.3].

**Proposition 5.1.** Given a non-decreasing function \( F : \mathbb{R} \to \mathbb{R} \) and a measurable function \( f : \mathbb{R} \to \mathbb{R} \) as above, it holds
\[ \int_{\mathbb{R}} F(f) dx = \int_{\mathbb{R}} F(f^*) dx. \]
The following Pólya-Szegő type inequality can be found e.g. in [18] (Inequality (3.6)) or [27].

**Theorem 5.2.** Let \( u \in H^{s,2}(\mathbb{R}) \) for \( 0 < s < 1 \). Then
\[
\int_{\mathbb{R}} |(-\Delta)^s |u|^2 dx \leq \int_{\mathbb{R}} |(-\Delta)^s u|^2 dx.
\]

Now given \( u \in H^{1/2,2}(\mathbb{R}) \), from Proposition 5.1, we get
\[
\int_{\mathbb{R}} (e^{\pi u^2} - 1) dx = \int_{\mathbb{R}} \left(e^{\pi (|u|^*)^2} - 1\right) dx, \quad \|u^*\|_{L^2} = \|u\|_{L^2},
\]
and according to Theorem 5.2
\[
\|u^*\|^2_{H^{1/2,2}(\mathbb{R})} = \|u^*\|^2_{L^2(\mathbb{R})} + \int_{\mathbb{R}} |(-\Delta)^{1/2} |u^*|^2 dx
\leq \|u\|^2_{L^2(\mathbb{R})} + \int_{\mathbb{R}} |(-\Delta)^{1/2} u|^2 dx = \|u\|^2_{H^{1/2,2}(\mathbb{R})}.
\]

Therefore, in the rest of the proof of (1.21), we may assume that \( u \in H^{1/2,2}(\mathbb{R}) \) is even, non-increasing on \([0, \infty)\), and \( \|u\|_{H^{1/2,2}(\mathbb{R})} \leq 1 \).

We write
\[
\int_{\mathbb{R}} (e^{\pi u^2} - 1) dx = \int_{\mathbb{R}\setminus I} (e^{\pi u^2} - 1) dx + \int_I (e^{\pi u^2} - 1) dx =: (I) + (II),
\]
where \( I = (-1/2, 1/2) \). We start by bounding (I). By monotone convergence
\[
(I) = \sum_{k=1}^\infty \int_{I} \pi^k u^2_k \frac{2^k}{k!} dx.
\]
Since \( u \) is even and non-increasing, for \( x \neq 0 \), we have
\[
u^2(x) \leq \frac{1}{2|x|} \int_{|x|} u^2(y) dy \leq \frac{\|u\|^2_{L^2}}{2|x|}, \tag{5.1}
\]
hence for \( k \geq 2 \), we bound
\[
\int_{I} u^2_k dx \leq 2^{1-k} \|u\|^2_{L^2(\mathbb{R})} \int_{I} \frac{1}{x^k} dx = \frac{\|u\|^2_{L^2(\mathbb{R})}}{(k-1)}.
\]
It follows that
\[
\sum_{k=2}^\infty \int_{I} \pi^k u^2_k \frac{2^k}{k!} dx \leq \sum_{k=2}^\infty \frac{(\pi \|u\|^2_{L^2})^k}{k!(k-1)}.
\]
Thus, since $\|u\|_{L^2(\mathbb{R})} \leq 1$, we estimate

$$(I) \leq \pi \|u\|^2_{L^2(\mathbb{R})} \left(1 + \sum_{k=1}^{\infty} \frac{(\pi \|u\|_{L^2(\mathbb{R})}^2)^k}{(k+1)!k} \right) \leq C.$$ 

We shall now bound $(II)$. We define the function $v : \mathbb{R} \to \mathbb{R}$ as follows

$$v(x) = \begin{cases} 
  u(x) - u(\frac{1}{2}) & \text{if } |x| \leq \frac{1}{2} \\
  0 & \text{if } |x| > \frac{1}{2}.
\end{cases}$$

Then with (5.1) and the estimate $2a \leq a^2 + 1$, we find

$$u^2 \leq v^2 + 2vu(\frac{1}{2}) + u(\frac{1}{2})^2 \leq v^2 + 2v\|u\|_{L^2(\mathbb{R})} + \|u\|_{L^2(\mathbb{R})}^2 \quad (5.2)$$

$$\leq v^2 + v\|u\|^2_{L^2(\mathbb{R})} + 1 + \|u\|^2_{L^2(\mathbb{R})} \leq v^2 \left(1 + \|u\|^2_{L^2(\mathbb{R})}\right) + 2.$$ 

Now, recalling that $u$ is decreasing, we have for $x \in I = [-\frac{1}{2}, \frac{1}{2}]$

$$\int_{\mathbb{R}} \frac{(v(x) - v(y))^2}{(x-y)^2} dy = \int_{I} \frac{(u(x) - u(y))^2}{(x-y)^2} dy + \int_{I^{c}} \frac{(u(x) - u(\frac{1}{2}))^2}{(x-y)^2} dy$$

$$\leq \int_{I} \frac{(u(x) - u(y))^2}{(x-y)^2} dy.$$ 

Notice that the last integral converges for a.e. $x \in I$ thanks to Proposition A.1 and Fubini’s theorem. Similarly for $x \in I^{c}$

$$\int_{\mathbb{R}} \frac{(v(x) - v(y))^2}{(x-y)^2} dy = \int_{I} \frac{(u(\frac{1}{2}) - u(y))^2}{(x-y)^2} dy$$

$$\leq \int_{I} \frac{(u(x) - u(y))^2}{(x-y)^2} dy \leq \int_{\mathbb{R}} \frac{(u(x) - u(y))^2}{(x-y)^2} dy.$$ 

Integrating with respect to $x$, we obtain

$$\|(-\Delta)^{\frac{1}{2}} v\|^2_{L^2(\mathbb{R})} = \frac{1}{C_s^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(v(x) - v(y))^2}{(x-y)^2} dy dx$$

$$\leq \frac{1}{C_s^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(u(x) - u(y))^2}{(x-y)^2} dy dx = \|(-\Delta)^{\frac{1}{2}} u\|^2_{L^2(\mathbb{R})},$$

where $C_s$ is as in Proposition A.1 below. Thus, since $\|u\|_{H^1} \leq 1$,

$$\|(-\Delta)^{\frac{1}{2}} v\|^2_{L^2(\mathbb{R})} \leq \|(-\Delta)^{\frac{1}{2}} u\|^2_{L^2(\mathbb{R})} \leq 1 - \|u\|^2_{L^2(\mathbb{R})}.$$
Therefore, if we set $w = v \sqrt{1 + \|u\|_{L^2(\mathbb{R})}^2}$, we have
\[
\|(-\Delta)^{\frac{1}{4}} w\|_{L^2(\mathbb{R})}^2 \leq \left(1 + \|u\|_{L^2(\mathbb{R})}^2\right) \left(1 - \|u\|_{L^2(\mathbb{R})}^2\right) \leq 1,
\]
hence, using the Moser-Trudinger inequality on the interval $I = (-1/2, 1/2)$ (Theorem 1.1), one has
\[
\int_I e^{\pi w^2} \, dx < C,
\]
and using (5.2)
\[
\int_I e^{\pi u^2} \, dx \leq e^{2\pi} \int_I e^{\pi w^2} \, dx \leq C,
\]
which completes the proof of (1.21).

It remains to prove (1.23). Given $\tau > 2$ consider the function
\[
f = f_\tau := \frac{1}{2\tau \sqrt{|x|}} \chi_{\{x \in \mathbb{R} : r < |x| < \delta\}}, \quad \delta := \frac{1}{2\tau}, \quad r := \frac{1}{2\tau^{\frac{1}{2}}}.
\]
Notice that $\|f\|_{L^2(\mathbb{R})}^2 = (2\tau)^{-1}$. Fix a smooth even function $\psi : \mathbb{R} \to [0, 1]$ with $\psi \equiv 1$ in $[-\frac{1}{2}, \frac{1}{2}]$ and $\text{supp}(\psi) \subset (-1, 1)$. For $x \in \mathbb{R}$, we set $u(x) = \psi(x)(F_{\frac{1}{4}} \ast f)(x)$, where $F_{\frac{1}{4}}(x) = (2\pi |x|)^{-\frac{1}{2}}$ is as in Lemma 2.1. Clearly $u \equiv 0$ in $\mathbb{R} \setminus I$, and $u$ is non-negative and even everywhere.

In the rest of the proof $s = \frac{1}{4}$. Notice that $(-\Delta)^s(F_{\frac{1}{4}} \ast f) = f$. This follows easily from Lemma 2.1 and the properties of the Fourier transform, see e.g. [19, Corollary 5.10]. Then, we compute
\[
(-\Delta)^s u = f + (-\Delta)^s[(\psi - 1)(F_{\frac{1}{4}} \ast f)] =: f + v,
\]
and set $g(x, y) = (\psi - 1)(x)F_{\frac{1}{4}}(x - y)$. Notice that $g$ is smooth in $\mathbb{R} \times (-\frac{1}{2}, \frac{1}{2})$.

We write
\[
v(x) = (-\Delta)^s \int_{\mathbb{R}} g(x, y) f(y) \, dy = \int_{\{r < |y| < \delta\}} (-\Delta)^s g(x, y) f(y) \, dy,
\]
where we used Proposition A.7 and Fubini’s theorem. With Jensen’s inequality
\[
\|v\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} \left| \int_{\{r < |y| < \delta\}} (-\Delta)^s g(x, y) f(y) \, dy \right|^2 \, dx \leq 2(\delta - r) \int_{\{r < |y| < \delta\}} f(y)^2 \int_{\mathbb{R}} |(-\Delta)^s g(x, y)|^2 \, dx \, dy
\]
\[
\leq 2\delta \|f\|_{L^2(\mathbb{R})}^2 \sup_{y \in [r, \delta]} \int_{\mathbb{R}} |(-\Delta_x)^s g(x, y)|^2 dx \leq C(\delta \tau^{-1}) = O(\tau^{-2}),
\]
where we used that
\[
\sup_{y \in [r, \delta]} \int_{\mathbb{R}} |(-\Delta_x)^s g(x, y)|^2 dx < \infty.
\]
This in turn can be seen noticing that \((-\Delta_x)^s g(x, y)\) is smooth, hence bounded on \([-R, R] \times [r, \delta]\) for every \(R\), and for \(|x|\) large and \(r \leq |y| \leq \delta\), using Proposition A.7
\[
(-\Delta_x)^s g(x, y) = C_s \int_R \frac{-F_s(x - y) - (\psi(z) - 1)F_s(z - y)}{|z - x|^{1+2s}} dz
- (-\Delta)^s F_s(x - y)
= O(|x|^{-1-2s}) \text{ uniformly for } |y| \leq \frac{1}{2},
\]
where we also used that \((-\Delta)^s F_s = 0\) away from the origin, see Lemma 2.1. Actually, with the same estimates, we get
\[
\int_{-\delta}^{\delta} |v|^2 dx \leq 2(\delta - r) \|f\|_{L^2(\mathbb{R})}^2 \int_{-\delta}^{\delta} \sup_{(x, y) \in [-\delta, \delta]^2} |(-\Delta_x)^s g(x, y)|^2 dx
\leq C\delta^2 \|f\|_{L^2(\mathbb{R})}^2 = O(\tau^{-3}).
\]
Therefore, using Hölder’s inequality and that \(\text{supp}(f) \subset [-\delta, \delta]\), we get, as \(\tau \to \infty\)
\[
\|(\Delta)^s u\|_{L^2(\mathbb{R})}^2 = \|f\|_{L^2(\mathbb{R})}^2 + \|v\|_{L^2(\mathbb{R})}^2 + 2 \int_{-\delta}^{\delta} f v dx = \frac{1}{2\tau} + O(\tau^{-2}). \tag{5.5}
\]
We now estimate \(u\). For \(0 < x < \tau\), with the change of variable \(\tilde{y} = \sqrt{\frac{x}{2}}\), we have
\[
u(x) = \frac{1}{2\tau \sqrt{2\pi}} \int_0^{\sqrt{\frac{x}{2}}} \left( \frac{1}{\sqrt{(y-x)y}} + \frac{1}{\sqrt{(y+x)y}} \right) dy
= \frac{1}{\tau \sqrt{2\pi}} \int_0^{\sqrt{x}} \left( \frac{1}{\sqrt{\tilde{y}^2 - 1}} + \frac{1}{\sqrt{\tilde{y}^2 + 1}} \right) d\tilde{y}
= \frac{1}{\tau \sqrt{2\pi}} \left( \log(\sqrt{\tilde{y}^2 - 1} + \tilde{y}) \right|_0^{\sqrt{x}} + \log(\sqrt{\tilde{y}^2 + 1} + \tilde{y}) \right|_0^{\sqrt{x}} \right)\]
\[
\frac{1}{\sqrt{2\pi}} + O(\tau^{-1}),
\]
with \(|\tau O(\tau^{-1})| \leq C\) as \(\tau \to \infty\) with \(C\) independent of \(x \in [0, r]\).

Similarly, for \(r < x < \delta\), we write
\[
\begin{align*}
    u(x) &\leq \frac{1}{\tau \sqrt{2\pi}} \left[ \int_r^x \frac{dy}{\sqrt{(x-y)y}} + \int_x^\delta \frac{dy}{\sqrt{(x-y)y}} \right] \\
    &= \frac{2}{\tau \sqrt{2\pi}} \left[ \int_1^\tau \frac{d\tilde{y}}{\sqrt{1-\tilde{y}^2}} + \log(\sqrt{\tilde{y}^2 - 1} + \tilde{y}) \bigg|_{1}^{\tau} \right] \\
    &= \frac{1}{\tau \sqrt{2\pi}} \left[ \log \left( \frac{\delta}{x} \right) + O(1) \right],
\end{align*}
\]

since \(\int_0^1 \frac{d\tilde{y}}{\sqrt{1-\tilde{y}^2}} < \infty\). Here, \(|O(1)| \leq C\) as \(\tau \to \infty\) with \(C\) independent of \(x \in (r, \delta)\).

When \(\delta < x < 1\) similar to the previous computation, and recalling that \(0 \leq \psi \leq 1\),
\[
\begin{align*}
    u(x) &\leq \frac{1}{\tau \sqrt{2\pi}} \int_r^\delta \frac{dy}{\sqrt{(x-y)y}} = \frac{2}{\tau \sqrt{2\pi}} \int_1^\tau \frac{d\tilde{y}}{\sqrt{1-\tilde{y}^2}} \\
    &\leq \frac{2}{\tau \sqrt{2\pi}} \int_0^1 \frac{d\tilde{y}}{\sqrt{1-\tilde{y}^2}} = O(\tau^{-1}),
\end{align*}
\]

with \(|\tau O(\tau^{-1})| \leq C\) as \(\tau \to \infty\) with \(C\) independent of \(x \in (0, 1)\). Thus
\[
\begin{cases}
    u(x) = \frac{1}{2\pi} + O(\tau^{-1}) & \text{for } 0 < x < r \\
    u(x) \leq \frac{2}{\tau \sqrt{2\pi}} \log \left( \frac{\delta}{x} \right) + O(\tau^{-1}) & \text{for } r < x < \delta \\
    u(x) = O(\tau^{-1}) & \text{for } \delta < x < 1.
\end{cases}
\]

Of course the same bounds hold for \(x < 0\) since \(u\) is even.

We now want to estimate \(\|u\|_{L^2(\mathbb{R})}^2\). We have
\[
\int_0^r u^2 \, dx = \tau \left( \frac{1}{2\pi} + O(\tau^{-1}) \right) = O(\tau^{-2}).
\]

For \(x \in [r, \delta]\), we have from (5.6)
\[
u(x)^2 \leq \frac{C}{\tau^2} \left( \log^2 \left( \frac{\delta}{x} \right) + \log \left( \frac{\delta}{x} \right) + 1 \right) \leq \frac{2C'}{\tau^2} \left( \log^2 \left( \frac{\delta}{x} \right) + 1 \right).
\]
Then, since
\[ \int_r^\delta \log^2 \left( \frac{\delta}{x} \right) \, dx = x \left( \log^2 \left( \frac{\delta}{x} \right) + 2 \log \left( \frac{\delta}{x} \right) + 2 \right) \bigg|_r^\delta \leq 2\delta = O(\tau^{-1}), \]
we bound
\[ \int_r^\delta u^2 \, dx = O(\tau^{-3}). \]

Finally, still using (5.6),
\[ \int_1^{\delta^2} u^2 \, dx = O(\tau^{-2}). \]

Also considering (5.5), we conclude
\[ \|u\|_{W^{s,2}([0,1])} = 2\|u\|_{L^2([0,1])} = O(\tau^{-2}), \quad \|u\|_{H^{s,2}(\mathbb{R})} = \frac{1}{2\tau} + O(\tau^{-2}). \quad (5.7) \]
Setting \( w_\tau := u\|u\|_{H^{s,2}(\mathbb{R})}^{-1} \), and using (5.6) and (5.7), we conclude
\[ \int_{-r}^r |w_\tau|^2 \left( e^{\pi w_\tau^2} - 1 \right) \, dx \geq \int_{-r}^r \left( \frac{\tau + O(1)}{\pi} \right) \left( e^{\tau+O(1)} - 1 \right) \, dx \geq \frac{r\tau e^\tau}{C} = \frac{1}{C}, \]

therefore,
\[ \lim_{\tau \to \infty} \int_{\mathbb{R}} h(w_\tau) \left( e^{\pi w_\tau^2} - 1 \right) \, dx \geq \int_{-r}^r h(w_\tau) \left( e^{\pi w_\tau^2} - 1 \right) \, dx \to \infty, \]
as \( \tau \to \infty \), for any \( h \) satisfying (1.22). \( \square \)

**Appendix A. Some useful results**

We define
\[ W^{s,p}(\mathbb{R}) := \left\{ u \in L^p(\mathbb{R}) : [u]_{W^{s,p}(\mathbb{R})}^p := \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(x) - u(y)|^p}{|x-y|^{1+sp}} \, dx \, dy < \infty \right\}. \quad (A.1) \]

**Proposition A.1.** For \( s \in (0,1) \), we have, \([u]_{W^{s,2}(\mathbb{R})} < \infty\) if and only if \((-\Delta)^{ \frac{s}{2} } u \in L^2(\mathbb{R})\), and in this case
\[ [u]_{W^{s,2}(\mathbb{R})} = C_s \|(-\Delta)^{ \frac{s}{2} } u\|_{L^2(\mathbb{R})}, \]
where \([u]_{W^{s,2}(\mathbb{R})}\) is as in (A.1) and \( C_s \) depends only on \( s \). In particular, \( H^{s,2}(\mathbb{R}) = W^{s,2}(\mathbb{R}) \).

**Proof.** See e.g. Proposition 3.6 in [12]. \( \square \)
Define the bilinear form
\[ B_s(u, v) = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{1+2s}} \, dx \, dy, \quad \text{for } u, v \in H^{s, 2}(\mathbb{R}), \]
where the double integral is well defined thanks to Hölder’s inequality and Proposition A.1.

The following simple and well-known existence result proves useful. A proof can be found (in a more general setting) in [14].

**Theorem A.2.** Given \( s \in (0, 1) \), \( f \in L^2(I) \) and \( g : \mathbb{R} \rightarrow \mathbb{R} \) such that
\[ \int_I \int_{\mathbb{R}} \frac{(g(x) - g(y))^2}{|x - y|^{1+2s}} \, dx \, dy < \infty, \tag{A.2} \]
there exists a unique function \( u \in \tilde{H}^{s, 2}(I) + g \) solving the problem
\[ B_s(u, v) = \int_{\mathbb{R}} f v \, dx \quad \text{for every } v \in \tilde{H}^{s, 2}(I). \tag{A.3} \]
Moreover such \( u \) satisfies \((-\Delta)^s u = \frac{C_s}{2} f\) in \( I \) in the sense of distributions, i.e.,
\[ \int_{\mathbb{R}} u(-\Delta)^s \varphi \, dx = \frac{C_s}{2} \int_{\mathbb{R}} f \varphi \, dx \quad \text{for every } \varphi \in C^\infty_c(I), \tag{A.4} \]
where \( C_s \) is the constant in Proposition A.7.

The following version of the maximum principle is a special case of Theorem 4.1 in [14].

**Proposition A.3.** Let \( u \in \tilde{H}^{s, 2}(I) + g \) solve (A.3) for some \( f \in L^2(I) \) with \( f \geq 0 \) and \( g \) satisfying (A.2) and \( g \geq 0 \) in \( I^c \). Then \( u \geq 0 \).

**Proof.** From Proposition A.1 it easily follows \( u^- := \min \{u, 0\} \in \tilde{H}^{s, 2}(I) \).
Then according to (A.3), we have
\[
0 \geq B_s(u, u^-) = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{u^+(x) + u^-(x) - u^+(y) - u^-(y))(u^-(x) - u^-(y))}{|x - y|^{1+2s}} \, dx \, dy
\]
\[
= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{u^-(x) - u^-(y))^2}{|x - y|^{1+2s}} \, dx \, dy - 2 \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{u^+(x)u^-(y)}{|x - y|^{1+2s}} \, dx \, dy,
\]
where we used that \( u^+u^- = 0 \). Since the second term in the last equality is non-negative, it follows at once that \( u^- \equiv 0 \), hence \( u \geq 0 \).
Proposition A.4. Let \( u \in \tilde{H}^{s,2}(I) \) be as in Theorem A.2 (with \( g = 0 \)), where we further assume \( f \in L^\infty(I) \). Then
\[
|u(x)| \leq C \|f\|_{L^\infty(I)} (\text{dist}(x, \partial I))^s,
\]
for every \( x \in I \). In particular, \( u \) is bounded in \( I \) and continuous at \( \partial I \).

Proof. This proof is inspired from [29], where a much stronger result is proven, i.e., \( u/(\text{dist}(\cdot, \partial I))^s \in C^\alpha(I) \) for some \( \alpha > 0 \).

To prove the proposition, we assume as usual that \( I = (-1,1) \) and recall that \( w(x) := \begin{cases} (1-|x|^2)^s & \text{for } x \in (-1,1), \\ 0 & \text{for } |x| \geq 1, \end{cases} \)
belongs to \( \tilde{H}^{s,2}(I) \) and solves \( (-\Delta)^s w = \gamma_s \) for a positive constant \( \gamma_s \), in the sense of Theorem A.2, i.e., (A.3) holds with \( u = w \) and \( f \equiv \gamma_s \) (see e.g. [15]). Then
\[
-\gamma_s (-\Delta)^s w \leq \frac{(-\Delta)^s u}{\gamma_s} \leq \gamma_s (-\Delta)^s w,
\]
and Proposition A.3 gives at once
\[
-\frac{\|f\|_{L^\infty(I)}}{\gamma_s} \leq u \leq \frac{\|f\|_{L^\infty(I)}}{\gamma_s} \text{ in } I.
\]
We conclude noticing that \( 0 \leq w(x) \leq 2^s (\text{dist}(x, \partial I))^s \).

The following density result is known for an arbitrary domain in \( \mathbb{R}^n \). On the other hand, its proof is quite complex in such a generality, hence we provide a short elementary proof which fits the case of an interval.

Lemma A.5. For \( s \in (0,1) \) and \( p \in [1, \infty) \) the sets \( C^\infty_c(I) \) (\( I \subset \mathbb{R} \) is a bounded interval) is dense in \( \tilde{H}^{s,p}(I) \).

Proof. Without loss of generality, we consider \( I = (-1,1) \). Given \( u \in \tilde{H}^{s,p}(I) \) and \( \lambda > 1 \), set \( u_\lambda(x) := u(\lambda x) \). We claim that \( u_\lambda \to u \) in \( \tilde{H}^{s,p}(I) \) as \( \lambda \to 1 \). Indeed
\[
\|u_\lambda - u\|_{\tilde{H}^{s,p}(I)} = \|u - u_\lambda\|_{L^p(\mathbb{R})} + \|\lambda^s f_\lambda - f\|_{L^p(\mathbb{R})},
\]
where \( f = (-\Delta)^{s/2} u \) and \( f_\lambda(x) := f(\lambda x) \). Since \( f \in L^p(\mathbb{R}) \) it follows that \( \|\lambda^s f_\lambda - f\|_{L^p(\mathbb{R})} \to 0 \) as \( \lambda \to 1 \), since this is obviously true for \( f \in C^0(\mathbb{R}) \) with compact support, and for a general \( f \in L^p(\mathbb{R}) \) it can be proven by approximation in the following standard way. Given \( \varepsilon > 0 \) choose \( f_\varepsilon \in C^0(\mathbb{R}) \)
with compact support and \( \| f_\varepsilon - f \|_{L^p(\mathbb{R})} \leq \varepsilon \). Then by the Minkowski inequality
\[
\| \lambda^s f_\lambda - f \|_{L^p(\mathbb{R})} \leq \| \lambda^s f_\lambda - \lambda^s f_{\varepsilon, \lambda} \|_{L^p(\mathbb{R})} + \| \lambda^s f_{\varepsilon, \lambda} - f_\varepsilon \|_{L^p(\mathbb{R})} + \| f_\varepsilon - f \|_{L^p(\mathbb{R})}
\]
and it suffices to let \( \lambda \to 1 \) and \( \varepsilon \to 0 \). Similarly \( \| u - u_\lambda \|_{L^p(\mathbb{R})} \to 0 \) as \( \lambda \to 1 \).

Now, given \( \delta > 0 \) fix \( \lambda > 1 \) such that \( \| u_\lambda - u \|_{H^{s,p}(\mathbb{R})} < \delta \) and let \( \rho \) be a mollifying kernel, i.e., a smooth non-negative function supported in \( I \) with \( \int_I \rho dx = 1 \). Also set \( \rho_\varepsilon(x) := \varepsilon^{-1} \rho(\varepsilon^{-1} x) \). Then noticing that \( u_\lambda \) is supported in \( [-\lambda^{-1}, \lambda^{-1}] \subseteq I \), for \( \varepsilon > 0 \) sufficiently small, we have that \( \rho_\varepsilon * u_\lambda \in C^\infty_c(I) \). To conclude the proof notice that
\[
\rho_\varepsilon * u_\lambda \to u_\lambda \text{ in } \tilde{H}^{s,p}(I) \text{ as } \varepsilon \to 0,
\]
since
\[
(-\Delta)^{\frac{s}{2}}(\rho_\varepsilon * u_\lambda) = \rho_\varepsilon * (-\Delta)^{\frac{s}{2}} u_\lambda \to (-\Delta)^{\frac{s}{2}} u_\lambda \text{ in } L^p(\mathbb{R}) \text{ as } \varepsilon \to 0 \text{ and } \lambda \downarrow 1.
\]

Proposition A.6. Let \( I \subseteq \mathbb{R} \) be a bounded interval and \( s \in (0,1) \). Let \( u \in L_s(\mathbb{R}) \) satisfy
\[
(-\Delta)^s u \geq 0 \text{ in } I \quad \text{(i.e., } \langle u, (-\Delta)^s \varphi \rangle \geq 0 \text{ for every } \varphi \in C_c^\infty(I) \text{ with } \varphi \geq 0 \text{)}), \quad \text{for all } \varphi \geq 0 \text{ in } I^c \text{ and}
\]
\[
\liminf_{x \to \partial I} u(x) \geq 0. \quad \text{(A.5)}
\]
Then \( u \geq 0 \) in \( I \). More precisely, either \( u > 0 \) in \( I \), or \( u \equiv 0 \) in \( \mathbb{R} \).

Proof. This is a special case of Proposition 2.17 in [32].

Remark 5. The statement of Proposition 2.17 in [32] is slightly different, since it assumes \( u \) to be lower-semicontinuous in \( I \). On the other hand, lower semicontinuity inside \( I \) already follows from [32, Prop. 2.15]. What really matters is condition (A.5). That an assumption of this kind (possibly weaker) is needed follows for instance from the example of Lemma 3.2.4 in [1].

The following way of computing the fractional Laplacian of a sufficiently regular function is often used.
Proposition A.7. For an open interval $J \subset \mathbb{R}$, let $s \in (0, \frac{1}{2})$ and $u \in L^s(\mathbb{R}) \cap C^{0, \alpha}(J)$ for some $\alpha \in (2s, 1]$, or $s \in \left[\frac{1}{2}, 1\right)$ and $u \in L^s(\mathbb{R}) \cap C^{1, \alpha}(J)$ for some $\alpha \in (2s - 1, 1]$. Then $\langle (-\Delta)^s u \rangle_J \in C^0(J)$ and
\[
(-\Delta)^s u(x) = C_s \text{P.V.} \int_{\mathbb{R}} \frac{u(x) - u(y)}{|x - y|^{1+2s}} dy = C_s \lim_{\varepsilon \to 0} \int_{\mathbb{R}\setminus[x-\varepsilon,x+\varepsilon]} \frac{u(x) - u(y)}{|x - y|^{1+2s}} dy
\]
for every $x \in J$. This means that, for every $\varphi \in C_c^\infty(J)$,
\[
\langle (-\Delta)^s u, \varphi \rangle = C_s \int_{\mathbb{R}} \varphi(x) \text{P.V.} \int_{\mathbb{R}} \frac{u(x) - u(y)}{|x - y|^{1+2s}} dy dx.
\]

Proof. See e.g. [32, Prop. 2.4]

Lemma A.8. Let $\varphi_1 \in H = \dot{H}^{\frac{1}{2}, 2}(I)$ be an eigenfunction corresponding to the first eigenvalue $\lambda_1(I)$ of $(-\Delta)^{\frac{1}{2}}$ on $I$. Then $\varphi_1 > 0$ a.e. on $I$ or $\varphi_1 < 0$ a.e. on $I$ and the corresponding eigenspace has dimension 1.

Proof. Recall that the first eigenvalue $\lambda_1(I)$ can be characterized by minimizing the following functional
\[
F(u) = \frac{\|u\|^2_{H}}{\int_I u^2 dx},
\]
that is,
\[
\lambda_1(I) = \min_{u \in H \setminus \{0\}} F(u).
\]
On the other hand using Proposition A.1, we get that for any $u \in H$
\[
\|u\|^2_{H} = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(u(x) - u(y))^2}{(x - y)^2} dx dy \geq \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(x)| - |u(y)|}{(x - y)^2} dx dy = \|u\|^2_{H},
\]
(A.6)
hence, $F(|u|) \leq F(u)$, and $F(u) = F(|u|)$ if and only if $u$ is non-negative or non-positive. Therefore, if $F(\varphi_1) = \lambda_1$, then $\varphi_1$ does not change sign. Moreover, Theorem A.1 in [9] gives us $\varphi_1 > 0$ or $\varphi_1 < 0$ almost everywhere in $I$. Any other eigenfunction corresponding to $\lambda_1$ must also have fixed sign, hence it cannot be orthogonal to $\varphi_1$, therefore it is a multiple of $\varphi_1$.

Lemma A.9. Consider a sequence $(f_k) \subset L^1(I)$ with $f_k \to f$ a.e. and with
\[
\int_{\{f_k > L\}} f_k dx = o(1), \quad (A.7)
\]
with $o(1) \to 0$ as $L \to \infty$ uniformly with respect to $k$. Then $f_k \to f$ in $L^1(I)$. 
Proof. From the dominated convergence theorem
\[ \min\{f_k, L\} \to \min\{f, L\} \text{ in } L^1(I), \]
and the convergence of \( f_k \) to \( f \) in \( L^1 \) follows at once from (A.7) and the triangle inequality. \( \square \)

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References