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Research Article

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Dynamics of the Third Exotic Contact Form on the Sphere Along a Vector Field in its Kernel

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Abstract: In this paper we study the dynamics of the third exotic contact form of Gonzalo and Varela [5] on S^3 in the framework of contact form geometry. This dynamics has an impact on the variational problem associated to the contact form such as the violation of the Fredholm condition and the Morse indices of different periodic orbits.

Keywords: Exotic Contact Structures, Morse Index, Dynamics

MSC 2010: 57R17, 37E35

Dedicated to the memory of Abbas Bahri, a great mathematician and even greater man

1 Introduction and Main Results

Let M be a three-dimensional closed compact manifold and α a contact form. That is, $\alpha \wedge d\alpha$ is a volume form on M. In order to investigate the existence of periodic orbits of the Reeb vector field, one is led to study the variations of the functional J defined on the loop space of the manifold M and given by

$$J(x) = \int_0^1 \alpha_{x(t)}(\dot{x}(t))\,dt.$$

Indeed, one can check that the critical points of the functional correspond to periodic orbits. The issue here is that studying the functional on the full loop space is not easy since the functional does not control the parts of the curve that are tangent to the kernel of α. Another issue is the fact that the functional is strongly indefinite on that space and the Morse index and co-index of every critical point are infinite. The choice of the space of variations is important in this case. Bahri [1] introduced a smaller space of loops, $\mathcal{C}_\beta$, making the study of the variations easier. In fact, the restriction of the functional to $\mathcal{C}_\beta$ has only the periodic orbits of the Reeb vector field as critical points. Moreover, the Morse index is finite and the difference of Morse indices in $\mathcal{C}_\beta$ is the same as in the free loop space. Also, it was proved in a previous work, see [7], that under mild assumptions there is an S^1-equivariant homotopy equivalence between the free loop space and $\mathcal{C}_\beta$. Therefore, from a Morse theoretical perspective, the study of the periodic orbits of the Reeb vector field, seems natural with the functional J restricted to the space $\mathcal{C}_\beta$. However, another unusual phenomena appears in this study that makes the application of the classical Morse theory approach much harder. That is, the violation of the Fredholm condition. Roughly speaking, the linearised operator of the action functional on the tangent space to $\mathcal{C}_\beta$ does not have the form $T + K$, where T is bicontinuous and K is compact. Hence, the classical variational theory does not apply. This fact from functional analysis has the geometric consequence presented in Figure 2.

For the sake of exposition, let us assume in a first case that the form α admits a Legendre transform, that is, there exists a non-vanishing vector field v in the kernel of α such that the 1-form $\beta = d\alpha(v, \cdot)$ is a contact

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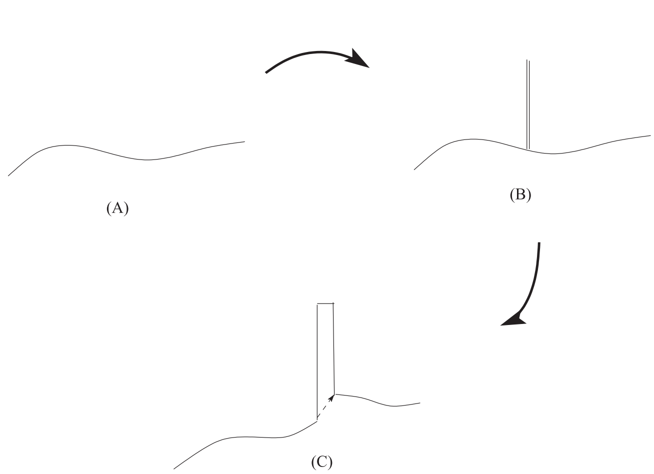

Figure 1. Violation of the Fredholm condition.

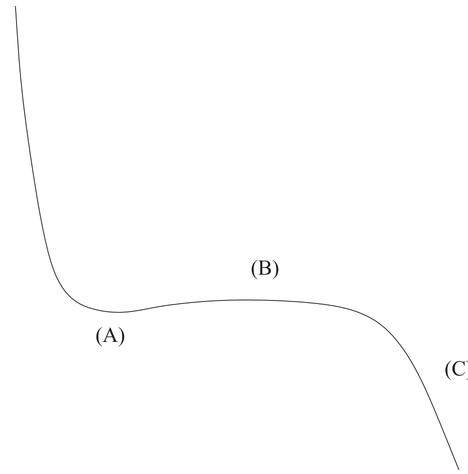

Figure 2. Geometric interpretation.

form with the same orientation. Such contact forms with their associated v exist in different settings and this condition is indeed equivalent to a certain convexity, see, for instance, [8–10]. In this setting (see [2]), we take our space of variations the set $\mathcal{C}_\beta$, defined by

$$\mathcal{C}_\beta = \{x \in H^1(S^1, M) : \beta(\dot{x}) = 0,\ \alpha(\dot{x}) = c > 0\},$$

where c is a non prescribed constant.

Bahri [2] proved that the Fredholm condition is violated at a given point of the curve if there exists $s \in \mathbb{R}$ such that

$$\alpha_{\varphi_{-s}}(D\varphi_s(\xi)) > 1.$$

In fact, this follows from the expansion of the functional J near a critical point with a back and forth v-run inserted on it (see Figure 1). If we call x_ε the new perturbed curve, we have that

$$J(x_\varepsilon) = J(x) + \varepsilon(1 - \alpha_{\varphi_{-s}}(D\varphi_s(\xi))) + o(\varepsilon).$$

Hence, if $\alpha_{\varphi_{-s}}(D\varphi_s(\xi)) > 1$, we have an extra decreasing direction. This can be thought of as presented in Figure 2. Part (A) represent the variation without the "Dirac mass" inserted. The critical point is "genuine", for example, a minimum. Part (B) corresponds to the insertion of the Dirac mass. The functional increases a tiny bit, as small as we wish, with the opening of the Dirac mass. It is zero at the Dirac mass. In the last part (C), the Dirac mass has reached a certain "height", that is, the back and forth or forth and back run along v has an appropriate size. "Opening" the Dirac mass, that is, increasing the length of the ξ-piece inserted between the vertical v-portions (see Figure 1), lowers the functional between the critical level.

There are other cases where the Fredholm condition is much harder to exhibit, especially for circle bundles. For example, one can check that all the tight contact forms of the torus T^3 admit a Legendre transform and $\alpha_{\varphi_{-s}}(D\varphi_s(\xi)) \leq 1$, but there exists $s \neq 0$ such that the equality holds, see [6].

The objective of the present paper is to study the sequence of overtwisted contact structures on S^3 introduced by Gonzalo and Varela [5], defined for $n \geq 0$ by

$$\alpha_n = -\Big(\cos\Big(\frac{\pi}{4} + n\pi(x_3^2 + x_4^2)\Big)(x_2 dx_1 - x_1 dx_2) + \sin\Big(\frac{\pi}{4} + n\pi(x_3^2 + x_4^2)\Big)(x_4 dx_3 - x_3 dx_4)\Big).$$

This sequence exhaust all the contact structures on the sphere, and for $n \geq 1$ these contact structures are overtwisted. In fact, we will study a toy model of these structures for the case $n = 3$. This study gives an idea on the behaviour of these contact forms for n odd. The even case is slightly different. There is a very important feature concerning the first exotic form, $n = 1$. In fact, for this form, there exists a vector field v for which the Legendre duality holds. This vector field was exhibited by Martino in [9], and in this setting, the Fredholm violation and the full dynamics were studied by Bahri in [4]. In our case, the vector field v that we consider (again introduced by Martino [9]) does not induce a Legendre transform and this gives a new feature to our paper.

In the first part of the paper we will study the dynamics of the contact form α_3 along the vector v in its kernel. From this study we deduce that the Fredholm condition is not satisfied.

Theorem 1.1. *There exist three tori s^+, s^- and T_0 such that every point in $S^3 - (s^+ \cup s^- \cup T_0 \cup C)$ is in A^+ and in A^-, where here C is the set of critical points of the total rotation function R.*

The sets $A^\pm$ introduced in this theorem represent the parts where one can decrease the functional by introducing a positive back and forth v-jump or a negative one.

After proving this theorem we move in the next part to exhibiting a foliation transverse to α stuck between the contact form α and its Legendrian dual β in the region where the later one is oriented negatively.

Theorem 1.2. *There exist $\delta > 0$ and a function F defined on the set $[\mathcal{P} \leq \delta]$ such that $dF(v) = 0$ and $dF(\xi) > 0$ for every $x \in [\mathcal{P} < 0]$.*

The last part of the paper deals with the study of the periodic orbits of the Reeb vector field from a variational point of view. That is, we compute the Morse indices of the orbits and their iterates, and their corresponding critical values.

Theorem 1.3. *The following relations hold:*

- *For $r_2 \neq 0, 1$, the Morse index of a simple periodic orbit takes the form*
$$i = 2(p - q),$$
where p and q are related by
$$\frac{\tilde{A}}{\tilde{B}} = \frac{p}{q}.$$
- *The critical values are related to the Morse index by*
$$c = \pi i \frac{h}{\tilde{A} - \tilde{B}}.$$
- *For $r_2 = 0$, the periodic orbit i_{O_0} satisfies*
$$19 \leq i_{O_0} \leq 20,$$
and for its iterate, we have
$$19k \leq i^k_{O_0} \leq 20k.$$

2 Definition and First Properties

In what follows, we will consider S^3 as a sub-manifold of $\mathbb{R}^4$ with the coordinate system (x_1, x_2, x_3, x_4). We will use the notation $r_1 = x_1^2 + x_2^2$ and $r_2 = x_3^2 + x_4^2$. Therefore, S^3 is defined by $r_1 + r_2 = 1$. We consider now the third Gonzalo–Varela form, defined by

$$\alpha = -\big(A_3(x_2 dx_1 - x_1 dx_2) + B_3(x_4 dx_3 - x_3 dx_4)\big),$$

where $A_3 = \cos(\frac{\pi}{4} + 3\pi r_2)$ and $B_3 = \sin(\frac{\pi}{4} + 3\pi r_2)$. We will first investigate the dynamics of a vector field on its kernel and then we will show that the Fredholm condition does not hold as well, similarly to the case $n = 1$, see [4].

Let $\tilde{A}_3 = A_3 + 3\pi r_1 B_3$ and $\tilde{B}_3 = B_3 + 3\pi r_2 A_3$ so that if we consider the vector field

$$\zeta = -(\tilde{B}_3 x_2 \partial_{x_1} - x_1 \partial_{x_2} + \tilde{A}_3 x_4 \partial_{x_3} - x_3 \partial_{x_4}),$$

then the Reeb vector field is $\xi = \frac{\zeta}{\alpha(\zeta)}$. For simplicity, we will use this notation for the following vector fields:

$$\begin{cases} x_2\partial_{x_1} - x_1\partial_{x_2} = \partial_{\theta_1}, \\ x_4\partial_{x_3} - x_3\partial_{x_4} = \partial_{\theta_2}, \\ x_1\partial_{x_1} + x_2\partial_{x_2} = \partial_{r_1}, \\ x_3\partial_{x_3} + x_4\partial_{x_4} = \partial_{\theta_1}. \end{cases}$$

With this notations, we define the following vector fields that will be used later in this paper:

$$X = \sqrt{2}\Big(\Big(\frac{B_3}{r_1}\partial_{\theta_1}\Big) - \Big(\frac{A_3}{r_2}\partial_{\theta_2}\Big)\Big),$$
$$Y = \frac{1}{r_1}\partial_{r_1} + \frac{1}{r_2}\partial_{r_2},$$
$$X_0 = \partial_{\theta_1} + \partial_{\theta_2}.$$

Also, define the two functions $a = x_1x_3 + x_2x_4$ and $b = x_1x_4 - x_2x_3$. Notice that $a = \sqrt{r_1r_2}\cos(\theta_1 - \theta_2)$ and $b = \sqrt{r_1r_2}\sin(\theta_2 - \theta_1)$. This latter remark is useful in proving the continuity of the desired vector field.

Lemma 2.1. *Using these notations, the following hold:*
(i) $[\zeta, X_0] = [X, X_0] = [Y, X_0] = 0$,
(ii) $X_0 \cdot a = X_0 \cdot b = 0$.

The vector fields X and Y generate the kernel of α whenever they are defined. But the vector field $v = aY + bX$ is in the kernel, globally defined and C^1. Another important information to deduce from the previous lemma is the fact that X_0 generates a symmetry in our setting since it commutes with everything.

Here, in order to define v, we used the notations of Martino [9], i.e., $v = aX + bY$. In this notation, a and b are not to be confound with the components of the tangent vector $\dot{x} = a\xi + bv$ to a curve x of $\mathcal{C}_\beta$. We will, in the sequel, be studying v and $\ker\alpha$ along v, so no confusion is possible. In what follows, a and b therefore refer to the components of v along X and Y.

Before initiating the study of the dynamics of v, we first write down the different differential relations between the quantities introduced above.

Proposition 2.2. *Using the previous notations, we have the following:*

$$\begin{aligned} &Y\cdot r_1 = 2, && Y\cdot r_2 = -2,\\ &\zeta\cdot a = -(\tilde{A}_3 - \tilde{B}_3)b, && \zeta\cdot b = (\tilde{A}_3 - \tilde{B}_3)a,\\ &Y\cdot a = \frac{r_2 - r_2}{r_2r_1}a, && Y\cdot b = \frac{r_2 - r_2}{r_2r_1}b,\\ &X\cdot a = -\sqrt{2}b\frac{A_3r_1 + B_3r_2}{r_1r_2}, && X\cdot b = \sqrt{2}a\frac{A_3r_1 + B_3r_2}{r_1r_2},\\ &Y\cdot\tilde{A}_3 = 6\pi(2B_3 - 3\pi r_1A_3), && Y\cdot\tilde{B}_3 = -6\pi(2A_3 - 3\pi r_2B_3),\\ &v\cdot r_1 = 2a, && v\cdot r_2 = -2a, \end{aligned}$$

and

$$v\cdot a = \frac{(r_2 - r_2)a^2}{r_2r_1} - \sqrt{2}b^2\frac{A_3r_1 + B_3r_2}{r_1r_2}.$$

It is important to notice here that v does not induce a Legendre transform for α_3. More precisely, we have the following proposition.

Proposition 2.3. *The* 1*-form* $\beta(\cdot) = d\alpha_3(v, \cdot)$ *is not a contact form.*

Indeed, an easy computation shows that

$$\beta\wedge d\beta(\xi, v, [\xi, v]) = \sqrt{2}(B_3\tilde{A}_3r_1 + A_3\tilde{B}_3r_2)(\tilde{A}_3 - \tilde{B}_3) + 6\pi a^2[\tilde{A}_3(3\pi r_1B_3 - 2A_3) - \tilde{B}_3(2B_3 - 3\pi r_2A_3)], \quad (2.1)$$

and as we can see in Figure 10, one can exhibit the region where it is negative in the (a, y)-plane.

3 Dynamics of *v*

The study of the dynamics of v along α_3 follows closely the analysis completed by Bahri for α_1. However, there are several differences when it comes to the violation of the Fredholm condition. New behaviour appears for α_3 making its study slightly more difficult compared to α_3.

From now on, we will write A for A_3 and B for B_3. We then write down the dynamical system generated by v as follows:

$$\begin{cases} \dot{x}_1 = \dfrac{a}{r_1}x_1 + \sqrt{2}\dfrac{bBx_2}{r_1}, \\ \dot{x}_2 = \dfrac{a}{r_1}x_2 + \sqrt{2}\dfrac{bBx_1}{r_1}, \\ \dot{x}_3 = -\dfrac{a}{r_2}x_3 - \sqrt{2}\dfrac{bAx_4}{r_2}, \\ \dot{x}_1 = \dfrac{a}{r_2}x_4 - \sqrt{2}\dfrac{bAx_3}{r_2}. \end{cases} \tag{3.1}$$

3.1 Evolution on the (a, y)-Variable

This system presents many symmetries that we can use to understand its evolution. Let $r_2 = y$. Then, we have the following:

$$v \cdot b = \frac{(2y-1) + \sqrt{2}(A(1-y) + By)}{y(1-y)}$$

and

$$\begin{cases} v \cdot y = -2a, \\ v \cdot a = \dfrac{((2y-1) + \sqrt{2}(A(1-y) + By))a^2}{y(1-y)} - \sqrt{2}(A(1-y) + By). \end{cases} \tag{3.2}$$

We define now the following quantities:

$$\begin{cases} f(y) = \dfrac{-1}{2}\dfrac{(2y-1) + \sqrt{2}(A(1-y) + By)}{y(1-y)}, \\ g(y) = (A(1-y) + By), \\ \quad p = -2a, \end{cases}$$

so that system (3.2) reads as follows:

$$\begin{cases} v \cdot y = p, \\ v \cdot p = f(y)p^2 + 2\sqrt{2}g(y), \end{cases} \tag{3.3}$$

and if we start from $b = b_0$, we have that along v,

$$b = b_0 e^{\int_{y_0}^{y} f(x)dx}.$$

Proposition 3.1. *Using the variable p and y we have the following:*
(i) *The system has three equilibrium points that can be identified by the value of* y, *two attractive* s^+, s^- *and one repulsive* $s_0 = \frac{1}{2}$.
(ii) *The orbits of this system are closed, except for the two homoclinic reaching* s_0 *in infinite time.*

Proof. Let us consider a function h, such that $h'(t) = e^{-\int_0^t f(s)ds}$. Let also $u = h(y)$. Then, it is easy to see that u satisfies

$$u'' = 2\sqrt{2}g(y)h'(y).$$

This reads also as

$$u'' = 2\sqrt{2}g(h^{-1}(u))h'(h^{-1}(u)) = F(u),$$

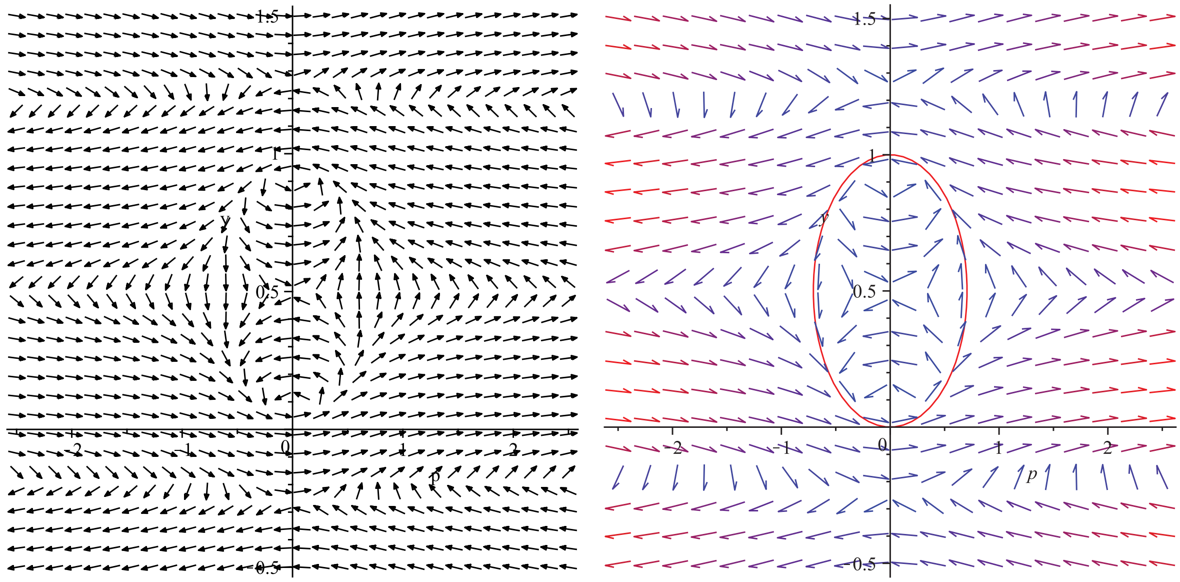

Figure 3. Flow of v.

which is a regular Newtonian second order dynamical system. From this fact one concludes that the following hold:

- All bounded orbits are closed, except the ones travelled in infinite time between unstable equilibriums.
- The equilibrium points in the phase plane correspond to $p = 0$ and zeros of g.

Hence, to get periodicity, it is enough to show that all the orbits in interest are bounded. Notice that we are only interested in the orbits located in the region corresponding to $0 < y < 1$. Indeed, since $p = \pm 2\sqrt{y(1-y)}$ satisfies (3.3) – and that is an ellipse, whose equation is $p^2 + 4(y - \frac{1}{2})^2 = 1$ – in the phase plane that contains all the region that we are interested in, all the orbits are bounded and hence periodic, except the ones mentioned before.

For the stability of the equilibrium values, it is enough to check the following determinant:

$$\begin{vmatrix} 0 & 2\sqrt{2}g'(y) \\ 1 & 2f(y)p \end{vmatrix},$$

and it is easy to check that $g'(y) = \tilde{B} - \tilde{A}$ and g has exactly three zeros in $[0, 1]$, which we order as $s^- < \frac{1}{2} < s^+$. The zeros are symmetric with respect to $\frac{1}{2}$. Also, s^- and s^+ are located in the region $\tilde{A} - \tilde{B} > 0$ and $\frac{1}{2}$ is located in the region $\tilde{A} - \tilde{B} < 0$, as shown in Figure 3, and this finishes the proof of the proposition. □

3.2 Total Rotation

Let us consider y as a variable of t along the v trajectories. We want to compute the total rotation along v. Let us use the following complex notation as in [4]. That is, we set

$$z = \frac{x_1 + ix_2}{\sqrt{r_1}}$$

and

$$z_1 = \frac{x_3 + ix_4}{\sqrt{r_2}}.$$

Therefore, system (3.1) becomes,

$$\begin{cases} \dot{z} = -i\dfrac{\sqrt{2}bB}{r_1}z, \\ \dot{z}_1 = i\dfrac{\sqrt{2}bA}{r_2}z_1. \end{cases} \tag{3.4}$$

Hence, we deduce that $z = e^{i\phi} z_0$ and $z_1 = e^{i\psi} z_{0,1}$, where

$$\phi(t) = -\int_0^t \frac{\sqrt{2}bB}{r_1} \quad \text{and} \quad \psi(t) = \int_0^t \frac{\sqrt{2}bA}{r_2}.$$

The total rotation R is then defined to be the difference between the values of ψ within a period of time, i.e.,

$$R(y) = \int_{T^-}^{T^+} \frac{\sqrt{2}bA}{r_2}.$$

We will have two formulae for R depending on the type of orbits.

Definition 3.2. We call orbits of type I, the orbits concentrating around s^+ or s^-, that is, the inner orbits like in Figure 3, and call orbits of type II, the ones crossing $\frac{1}{2}$, that is, the ones turning around type I orbits and bounding them.

3.2.1 Type I Orbits

Let us start by studying type I orbits. Consider for instance, an orbit associated to s^+. Call T^- and T^+ the first times before and after the orbit crosses s^+, starting from a crossing of s^+. Then,

$$R(y) = \int_{T^-}^{T^+} \frac{\sqrt{2}bA}{r_2},$$

and taking $y_M = \max y$ and knowing that $b = b_0 e^{\int_{y_0}^{y} f(x)dx}$, at $y = y_M$ we get

$$b = \sqrt{y_M(1-y_M)}e^{\int_{y_M}^{y} f(x)dx}.$$

Also, if we consider the function

$$k(y) = y(1-y)e^{-2\int_{1/2}^{y} f(x)dx},$$

then we have

$$b(y) = \sqrt{k(y_M)}e^{\int_{1/2}^{y} f(x)dx}$$

and

$$a(y) = \sqrt{k(y) - k(y_M)}e^{\int_{1/2}^{y} f(x)dx}.$$

So we can write

$$R(y_M) = \int_{T^-}^{T^+} \frac{\sqrt{2}bA}{-2ay}y' = \int_{y_m}^{y_M} \frac{\sqrt{2}\sqrt{k(y_M)}e^{\int_{1/2}^{y} f(x)dx}\cos(\frac{\pi}{4} + 3\pi y)}{-2\sqrt{k(y) - k(y_M)}e^{\int_{1/2}^{y} f(x)dx}y} = -\sqrt{2k(y_M)}\int_{y_m}^{y_M} \frac{\cos(\frac{\pi}{4} + 3\pi y)}{2y\sqrt{k(y) - k(y_M)}}\,dy,$$

where y_m is the minimal value of y in the orbit. Notice that y_M and y_m are related through k. In fact, looking at the graph of k we can see how they are related even though we do not have an explicit formula for it (see Figure 4). Hence, R depends only on y_M.

3.2.2 Type II Orbits

For type II orbits the formula is more explicit since there is more symmetries with respect to $y = \frac{1}{2}$ and hence, using a similar computation, we get

$$R(y_M) = -\sqrt{2k(y_M)}\int_{1-y_M}^{y_M} \frac{\cos(\frac{\pi}{4} + 3\pi y)}{2y\sqrt{k(y) - k(y_M)}}\,dy = -2\sqrt{2k(y_M)}\int_{1/2}^{y_M} \frac{\cos(\frac{\pi}{4} + 3\pi y)}{y\sqrt{k(y) - k(y_M)}}\,dy - 2\tan^{-1}\left(\sqrt{\frac{1}{4k(y_M) - 1}}\right).$$

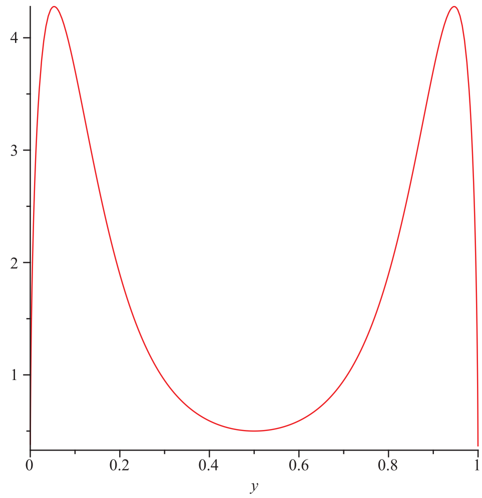

Figure 4. The graph of k.

4 Conjugate Points

First we recall that two points on an orbit of v are said to be conjugates if the contact plane makes a half rotation (π rotation) between those two points. In this case we say that the contact form turns well along v. This condition is very important in the study of the topology of the space $\mathcal{C}_\beta$, see [7]. We will distinguish the two cases depending on the number zeros of a between the crossings with s^+, that is, if it is even or odd.

4.1 The Even Case

Let us consider first type I orbits. Notice here that in this case b is never zero. We will assume as a first step that $a \neq 0$ at the conjugate points. Since X_0 is transported along v, at any two conjugate points $\alpha(X_0)$ is the same, but $\alpha(X_0) = g(y)$. Hence, two conjugate points must have the same image with g, that is, if we take a look at the graph of g (see Figure 5) we see that we have two cases, either they are on the same torus (when $y > s^+$) or they can be in two different tori (when $\frac{1}{2} < y < s^+$, in fact they coincide when y is a critical point of g, that is, when $\tilde{A} = \tilde{B}$).

Let us call $\overline{\psi}$ the transport map from a torus to itself, that is, it sends a point in the torus to the second intersection of the v orbit within it, as represented in Figure 6.

Lemma 4.1. *With this definition of $\overline{\psi}$, the following hold:*
(i) *The torus $y = s^+$ is a characteristic surface.*
(ii) *If z and $\overline{\psi}(z)$ are conjugate, then $d\overline{\psi}_z = \mathrm{id}$.*

Proof. Clearly, since X_0 is v-transported,

$$d\overline{\psi}(X_0) = X_0 + \mu v,$$

but since $a \neq 0$ and $\overline{\psi}$ is a map from the torus to itself, we have that $\mu = 0$, and by density we get

$$d\overline{\psi}(X_0) = X_0,$$

even when $a = 0$. So if $z \in T_r = [y = r]$, then its conjugate should be $\overline{\psi}(z)$. Now let us look at $d\overline{\psi}(X)$. Since z and $\overline{\psi}(z)$ are conjugate, the kernel is mapped to itself, that is, since $a \neq 0$,

$$d\overline{\psi}(X) = \theta X + \mu v,$$

and using the same argument as before, we have

$$d\overline{\psi}(X) = \theta X.$$

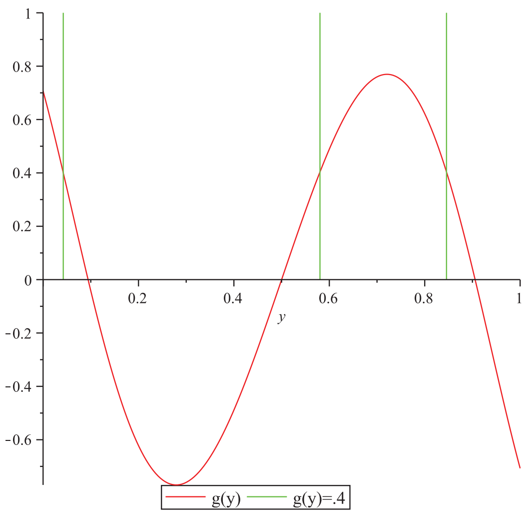

Figure 5. The graph of g.

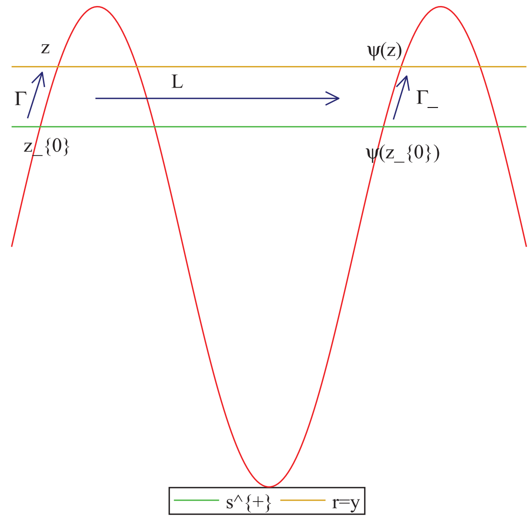

Figure 6. Conjugate points.

Now if we look at a, we have that since it depends on r along v, $a(z) = a(\overline{\psi}(z))$, and by taking the differential, we get $da_z = da_{\overline{\psi}(z)}(d\overline{\psi})$. Therefore,

$$da_z(X) = da_{\overline{\psi}(z)}(d\overline{\psi}(X)) = \theta da_{\overline{\psi}(z)}(X),$$

and assuming that b is not maximal, we get that $\theta = 1$ since $da_z(X) = da_{\overline{\psi}(z)}(X)$. So we have

$$d\overline{\psi}(X) = X.$$

Now we claim that

$$d\overline{\psi}_z(\zeta) = \zeta.$$

For this, we define Γ the differential of the transport along v from $r = s^+$ to $r = r_0$. Let L be the differential of the transport from $[z_0, z]$ to $[\overline{\psi}(z_0), \overline{\psi}(z)]$ and, to finish, let Γ_1 be the differential of the transport map from $\overline{\psi}(z_0)$ to $\overline{\psi}(z)$. Thus, we have

$$\Gamma_1 = L \circ \Gamma \circ L^{-1}$$

and

$$\begin{aligned} d\overline{\psi}_{z_0} &= \Gamma_1^{-1} \circ d\overline{\psi}_z \circ \Gamma \\ &= L \circ \Gamma^{-1} \circ L^{-1} \circ d\overline{\psi}_z \circ \Gamma. \end{aligned}$$

Taking into account that L is just a rotation and X and X_0 commute, we have

$$d\overline{\psi}_{z_0}(\Gamma^{-1}(X)) = L \circ \Gamma^{-1}(X).$$

Writing $\Gamma^{-1}(X) = \theta X_0 + \gamma\zeta$, where $\gamma \neq 0$, we have

$$d\overline{\psi}_{z_0}(\theta X_0 + \gamma\zeta) = L(\theta X_0 + \gamma\zeta).$$

Using again the fact that it is just a rotation, we get the claim.

Now it remains to show that s^+ is also a characteristic surface. Let us again consider the map $\overline{\psi}$ from s^+ to itself. We have that $\overline{\psi}$ sends X_0 to itself and of course v to itself since it is the 1-parameter group of v. So we need to check now what is the image of ζ. We have that $d\overline{\psi}(\zeta) = a_1 X_0 + b_1\zeta$. Since $a(z) = a(\overline{\psi}(z))$, we have

$$da_z(\zeta) = da_{\overline{\psi}(z)}(d\overline{\psi}(\zeta)).$$

Therefore,

$$-b(\tilde{A} - \tilde{B}) = -bb_1(\tilde{A} - \tilde{B}).$$

So if $b \neq 0$, we have that $b_1 = 1$ and $\alpha_z = \alpha_{\overline{\psi}(z)}(d\overline{\psi})$. □

4.1.1 Type II Orbits

For type II orbits we have the same results for conjugate points in the same torus. However, there is a difference from the previous case since if we take a look at the graph of g and consider the set $g(y) = c$, we find two cases. The first one corresponds to one torus and is similar to type I orbits. The other case, which we will investigate, is when there are three intersections. In the latter case, we have one torus bellow s^- and two tori between $\frac{1}{2}$ and s^+ (by symmetry we have a similar behaviour from the other side). So we have the following result.

Lemma 4.2. *For the map ψ defined above, the following hold:*
(i) *If z and $\psi(z)$ are conjugate, then $d\psi_z = \mathrm{id}$.*
(ii) *The torus $y = \frac{1}{2}$ is a characteristic surface.*
(iii) *The tori $y = s^+$ and $y = s^-$ are characteristic and conjugate to each other.*

Proof. Assertion (i) is similar to the previous type of orbits, so let us just check (ii) and (iii). The only part that one needs to check is what happens to the transport of ζ, but that depends only on how we write our map, since a is the same. Let us consider the case of Figure 5, and take the tori 1, 2 and 3. When 1 moves down toward $\tilde{A} = \tilde{B}$, torus 2 moves upward toward it and 3 moves to $y = 0$. □

4.2 The Odd Case

In this part we will exhibit conjugate points separated by an odd number of zeros of a. We will do the proof for the case of type I orbits. The type II case is similar, up to a small modification, because of the rotation formula.

Define the map $l\colon S^3 \to S^3$ by $l(x_1, x_2, x_3, x_4) = (x_3, x_4, -x_1, -x_2)$. It is in fact the action of the complex structure of $\mathbb{C}^2$. We consider also the map $\theta\colon s^+ \to s^+$ that maps the point A (the intersection of the v-orbit with s^+) to the point B, which is the next intersection (see Figure 7). The following lemma holds.

Lemma 4.3. *Assume that y_M is not a critical point of the total rotation R. Then, there exists $m > 0$ such that $d\overline{\psi}(\xi) = (1 + \gamma)\xi - m^2 X$, where $\gamma > 1$ on s^+.*

Proof. First recall that $\overline{\psi}(z_0, z) = (e^{iR(y_M)} z_0, e^{iR(y_M)} z)$, therefore

$$d\overline{\psi}(\xi) = \xi + iR_\xi \overline{\psi}(z_0, z).$$

We compute now the value of R_ζ. Recall that

$$R = \int_{T^-}^{T^+} \frac{\sqrt{2}bA}{r_2},$$

thus

$$R_\zeta = \int_{T^-}^{T^+} \frac{\sqrt{2}aA(\tilde{A} - \tilde{B})}{r_2} = -\frac{1}{2}\int_{T^-}^{T^+} \frac{\sqrt{2}A(\tilde{A} - \tilde{B})}{r_2} y' = -\frac{1}{2}\int_{y_m}^{y_M} \frac{\sqrt{2}A(y)(\tilde{A} - \tilde{B})}{y}\, dy$$

and this can be easily seen to be negative since $y_M > s^+$. Note that

$$i\overline{\psi}(z_0, z) = -\frac{1}{\sqrt{r_1}}\partial_{\theta_1} - \frac{1}{\sqrt{r_2}}\partial_{\theta_2} = c_1\zeta + c_0 X_0,$$

where

$$c_1 = \frac{\sqrt{r_2} - \sqrt{r_1}}{r_2 r_1(\tilde{A} - \tilde{B})} \quad \text{and} \quad c_0 = \frac{-1}{\sqrt{r_2}} + c_1\tilde{A}.$$

It is easy to check that at s^+, c_0 is a negative number and c_1 is a positive number. Therefore, we have $d\overline{\psi}(\xi) = (1 + \gamma)\xi - m^2 X$. □

Let A, B, C, D and E as described before. Take two points A' between D and A and C' between C and E.

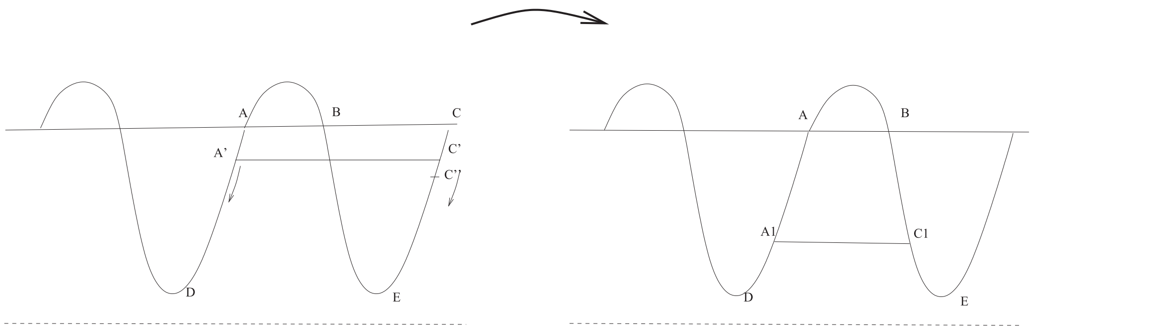

Figure 7. Conjugate points in the odd case.

Lemma 4.4. *The rotation between A' and C' is more than 2π.*

Proof. Consider the vector field $T_0 = X_0 - \alpha_{A'}(X_0)\xi$. We transport this vector to A, and take the time of transport to be δ. Since X_0 is transported, we need only to worry about the transport of ξ. So let us write down the transport equations of ξ along v in the (ξ, v, w), where w is a vector such that $\beta(w) \neq 0$. That is,

$$\begin{cases} \dot{\lambda} = \eta, \\ \dot{\eta} = d\beta(v, w)\eta - \lambda\mathcal{P}, \\ \lambda(0) = 1, \quad \eta(0) = 0. \end{cases}$$

Here $\mathcal{P} = *\beta \wedge d\beta$. Therefore, if we set

$$M(t) = \begin{pmatrix} 0 & 1 \\ -\mathcal{P} & d\beta(v, w) \end{pmatrix},$$

we have, using Gronwall's inequality,

$$\left\| Y(\delta) - Y(0) - \int_0^\delta MY(0)\, dt \right\| \leq C\delta^2 e^{\delta C} \quad \text{for } Y(t) = \begin{pmatrix} \lambda \\ \eta \end{pmatrix}.$$

Hence, the transported ξ at A reads as

$$\xi + C\delta\mathcal{P}w + O(\delta^2).$$

But $\mathcal{P}$ is positive in the neighbourhood of s^+. Thus, the transport of the vector T_0, that we call T_1, is

$$T_1 = X_0 - \alpha_{A'}(X_0)(\xi + C\delta w + O(\delta^2)).$$

We can take $w = X_0$ in the neighbourhood of s^+, since it is different from ξ. Thus, we can finally write T_1 as

$$\begin{aligned} T_1 &= (1 + C\delta\alpha_{A'}(X_0))X_0 - \alpha_{A'}(X_0)(\xi + O(\delta^2)) \\ &= (1 + C\delta\alpha_{A'}(X_0))\Big(X_0 - \frac{\alpha_{A'}(X_0)}{1 + C\delta\alpha_{A'}(X_0)}(\xi + O(\delta^2))\Big). \end{aligned}$$

Again we transport T_1 to C. Using (i), we have that the transport of T_1, denoted by T_2, reads as

$$\begin{aligned} T_2 &= (1 + C\delta\alpha_{A'}(X_0))\Big(X_0 - \frac{\alpha_{A'}(X_0)}{1 + C\delta\alpha_{A'}(X_0)}[(1+\gamma)\xi - m^2X_0 + O(\delta^2)]\Big) \\ &= (1 + C\delta\alpha_{A'}(X_0))(1 + \frac{\alpha_{A'}(X_0)}{1 + C\delta\alpha_{A'}}m^2)\Big[X_0 - \Big(\frac{\alpha_{A'}(X_0)(1+\gamma)}{(1 + C\delta\alpha_{A'}(X_0))(1 + \frac{\alpha_{A'}(X_0)}{1+C\delta\alpha_{A'}}m^2)}\Big)(\xi + O(\delta^2))\Big]. \end{aligned}$$

Since $\gamma > 0$ and $\alpha_{A'}(X_0) < 0$, we have that the ξ component of T_2 is larger than the one of T_1 for δ small. Hence, if we take the vector T at C' and transport it back to C, knowing that $\alpha_{C'}(X_0) = \alpha_{A'}(X_0)$, we will always have a smaller component along ξ compared to T_2. Hence, in order to have the same component, we should come from a point further than C' that is bellow C'. ☐

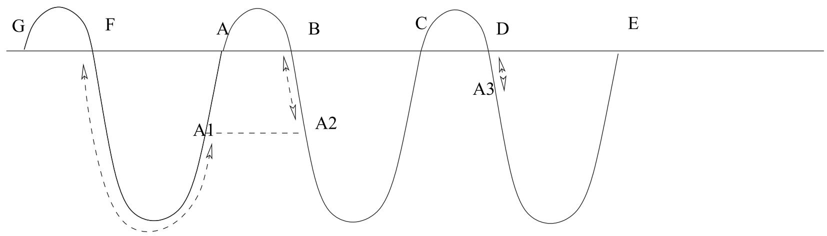

Figure 8. Transport along v and violation of Fredholm condition.

From this lemma we get that there exist two conjugate points separated by a zero of a, between the points D and E. To see this, we start moving A' back, thus the point C' will go backward and so does the point C''. If the point C' and C'' coincide along the way before crossing a zero of A, then this leads to a critical point of R which is rejected by assumption; the same holds if this happens at $a = 0$. Thus, by continuity, there exists a position r_0 for which A' and C'' are in the same torus $y = r_0$ and separated by just one zero.

5 The Fredholm Aspect

In this section we will use the results from the study of the dynamics of v to prove that indeed the Fredholm condition is violated.

Definition 5.1. If ϕ_s denote the one parameter group of v, we set

$$A^+ = \{x_0 \in M : \alpha_{x_0}(D\phi_s(\xi(x_{-s})) = \alpha_{x_0}(D\phi_s(\xi(D\phi_{-s}(x_0))) > 1 \text{ for a certain } s > 0\},$$

and in a similar way we define

$$A^- = \{x_0 \in M : \alpha_{x_0}(D\phi_s(\xi(x_{-s})) = \alpha_{x_0}(D\phi_s(\xi(D\phi_{-s}(x_0))) > 1 \text{ for a certain } s < 0\}.$$

So basically A^+ is the set of points from which the Fredholm condition is violated by a positive back and forth v-jump, and A^- is the set of points from which the Fredholm condition is violated by a negative back and forth v-jump.

The main result of this section can be stated as follows (compare to [4, Proposition 3 and 4]).

Proposition 5.2. *Every point in $S^3 - (s^+ \cup s^- \cup T_0 \cup C)$ is in A^+ and in A^-, where C is the set of critical points of the total rotation function R.*

Lemma 5.3. *All the points in type I orbits, except the characteristic tori and the critical points of R, belong to A^+ or A^-.*

The key here is to use the result of the previous section about the existence of conjugate points separated by an odd number of zeros of a.

Proof. Consider a type I orbit, as in Figure 8, such that y_m is not a critical point of R. For instance, we will assume that the total rotation between two points below the characteristic torus is more than 2π. Then, we know that we have two conjugate points A_1 and A_2. After a 2π rotation, the transport map along v will map F to B and A_1 to A_2. Hence, by the monotonicity of the rotation, we have that the portion $[F, A_1]$ is mapped to $[B, A_2]$. Now notice that starting from A_2, since B is mapped to D, we have that there exist a point in the interval $[D, E]$ where $\ker\alpha$ makes a 2π turn. Let us call that point A_3. We claim that A_3 is above the torus containing A_2. In fact, it cannot be exactly at the torus since y_m is not a critical point of R. So it is either below or above it. Assume it is below it. Then, by continuity we have that the rotation is less that 2π, which is impossible.

So now we have a point A_3 of coincidence for α. By iterating the process, we find a sequence of points A_k of coincidence of α such that they converge to the characteristic torus. Hence, if s_k the time corresponding

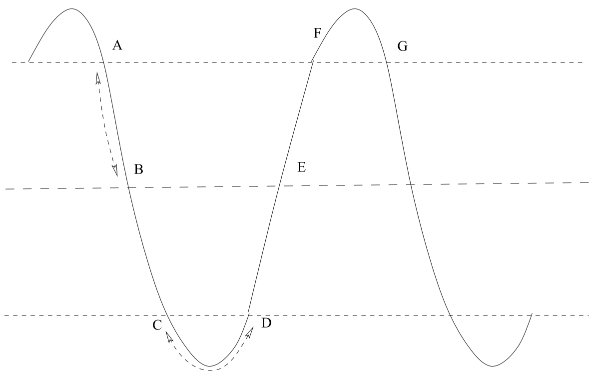

Figure 9. Transport for type II orbits

to the point A_k, we have $\varphi_{s_k}^*\alpha = \lambda_k\alpha$. Knowing that X_0 is transported along v, we have that λ_k converges to zero as k goes to infinity. Thus, starting from a point in $[F, A_1]$, by transport, we can make $\alpha_{x_0}(D\phi_s(\xi(x_{-s}))$ arbitrarily large, making it a point in A^+. To show that it is in A_-, it is enough to iterate the process in the other direction.

Notice that the proof is the same if we assume that the rotation is less that 2π. □

Lemma 5.4. *For type II orbits, the regions $y > s^+$ and $y < s^-$ are parts of A^+.*

One can follow the same procedure to get the result, but for parts of the v orbits above s^+ and below s^- and for rotation 4π instead of 2π. This is because the two tori s^+ and s^- are characteristic and conjugate. An important thing to notice here is the fact that if we try to apply the procedure for points between s^+ and s^-, then we get stopped by the two preceding conjugate tori and we cannot keep turning. Nevertheless, we have the following lemma.

Lemma 5.5. *Any point of a type II orbit is in A^+.*

Proof. What makes it work in this case is that the portion of orbit $[A, B]$ is mapped to $[C, D]$ as it is shown in Figure 9. In fact, since X_0 is transported along v and $\alpha(X_0) = 0$ exactly at the characteristic tori, then we have that the total rotation from A to C is 2π, by the monotonicity of the rotation of $\ker\alpha$. Now after mapping that portion to $[C, D]$, we are again in the region $y < s^-$ and using the 4π transport map as in Lemma 5.4, we obtain the result of the lemma. □

There is another important property that one can notice by studying the variations of a and b along ξ. If we call $h = \alpha(\zeta)$, then

$$\xi = \frac{1}{h}\zeta,$$

and we have the following identities.

Lemma 5.6. *If we set*

$$\tau = \frac{(\tilde{A} - \tilde{B})^2}{h^2},$$

then the following hold:
(i) $\xi\xi a = -\tau a$, $\xi\xi b = -\tau b$,
(ii) $[\xi, [\xi, v]] = -\tau v$.

Hence, the characteristic length is determined exactly by τ, which governs the same behaviour of a and b along ξ, and since ξ is always tangent to the tori $r = cte$, we have that τ is constant and thus a and b are linear combination of sines and cosines. Hence, in a piece of characteristic length we can reduce them to be as small as we can, and thus the part $|a| < c_0$ in the orbit will be in type I v orbits. Therefore, using the preceding procedure we get the Fredholm violation.

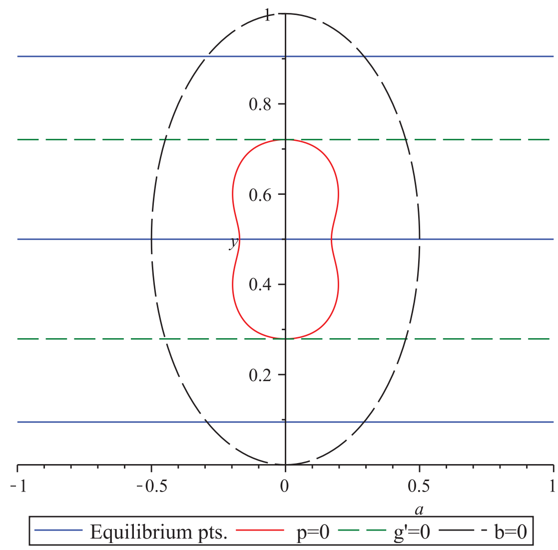

Figure 10. Different zones in the (a,y)-plane.

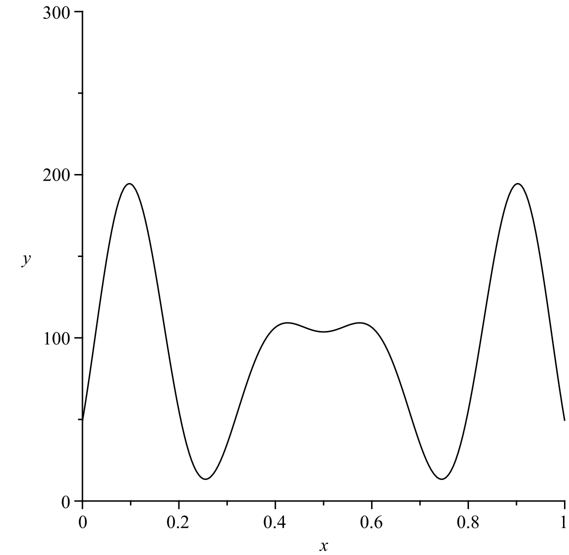

Figure 11. The value of $\mathcal{P}$ when $b = 0$.

6 The Function $\frac{a}{b}$

In the previous sections we saw that β changes its behaviour from a positively oriented contact form to a negatively oriented one (see Figure 10). We set $\mathcal{P} = -d\alpha(v, [\xi, v])$. The sign of $\mathcal{P}$ determines the behaviour of β. In particular, in the region $\mathcal{P} < 0$, we have two contact forms α and β that are transverse to each other and turn in opposite directions. So we expect the existence of a foliation stuck between them. In this section we will exhibit such foliation in the region $\mathcal{P} \leq \delta$ for δ small and positive. This can be stated as follows.

Proposition 6.1. *There exist $\delta > 0$ and a function F defined on the set $[\mathcal{P} \leq \delta]$ such that $dF(v) = 0$ and $dF(\xi) > 0$ for every $x \in [\mathcal{P} < 0]$.*

This supports the more general conjecture by Bahri [2] that there is a foliation γ transverse to α and β in the region $\beta \wedge d\beta < 0$.

Proof. This map will be constructed in several steps depending on the type of orbits and the range of y. First we will construct the function in the region $\{y > \frac{1}{2}\}$. We take the map F to be constant on the trajectories of v and equal to the value of $\frac{a}{b}$ at $y = s^+$. That is,

$$F(x) = \frac{a}{b}(\varphi_{t^+(x)}(x)) + \frac{a}{b}(\varphi_{t^-(x)}(x)),$$

where $t^+(x)$ (resp. $t^-(x)$) is the time needed for the v orbit to hit s^+ flowing with positive times (resp. flowing backward) as shown in Figure 12.

Now to show that this map is well defined, we proceed as follows. Assume first that $b = 0$, that is, $a^2 = y(1 - y)$. Replacing it in (2.1), we get the existence of $c > 0$ such that $\mathcal{P} > c$ (see Figure 11). So if we pick $\delta = \frac{c}{2}$, then the map is well defined. In fact, as stated before, $b = 0$ corresponds to the v orbit representing the ellipse containing all the other orbits (see Figure 10). Also from Proposition 2.3, we have, if $b = 0$,

$$\beta \wedge d\beta(\xi, v, [\xi, v]) = \sqrt{2}(B\tilde{A}r_1 + A\tilde{B}r_2)(\tilde{A} - \tilde{B}) + 6\pi r_2(1 - r_2)[\tilde{A}(3\pi r_1 B - 2A) - \tilde{B}(2B - 3\pi r_2 A)],$$

and this is represented in Figure 11.

Every point $x \in [\mathcal{P} < 0]$ is contained in an orbit crossing $s^\pm$, except of course the points in $r_2 = \frac{1}{2}$ with $a = 0$. By continuity, we assign to these points the value of points in the homoclinic orbit as they are reached in infinite time.

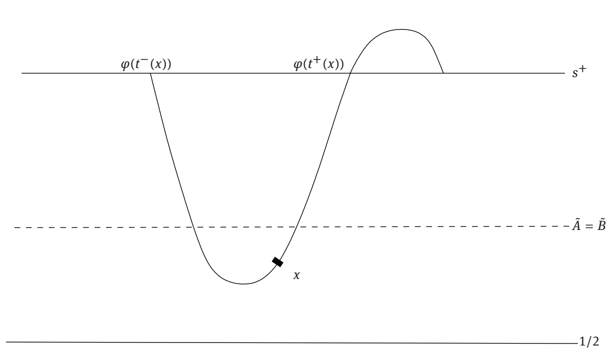

Figure 12. The map F for type I orbits.

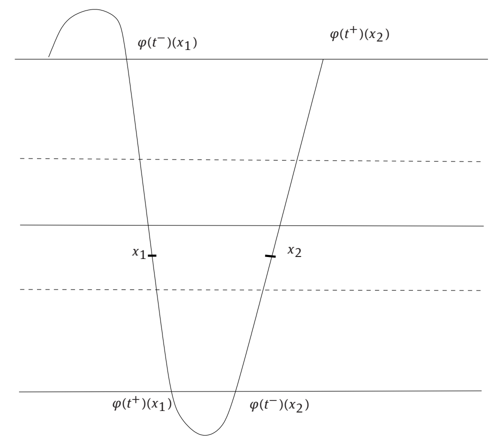

Figure 13. The map F for type II orbits.

Since F is constant along the orbits of v, $dF(v) = 0$. Now the only thing we need to check is $dF(\zeta) > 0$. Note that since F is defined using the transport along v, we have

$$\begin{aligned} dF(\zeta) &= d\Big(\frac{a}{b}\Big)(D\varphi_{\varphi_{t^+(x)}(x)}(\zeta) + dt^+(x)v) + d\Big(\frac{a}{b}\Big)(D\varphi_{\varphi_{t^-(x)}(x)}(\zeta) + dt^-(x)v) \\ &= d\Big(\frac{a}{b}\Big)(D\varphi_{\varphi_{t^+(x)}(x)}(\zeta)) + d\Big(\frac{a}{b}\Big)(D\varphi_{\varphi_{t^-(x)}(x)}(\zeta)). \end{aligned}$$

Writing $D\varphi_{t^+(x)}(\zeta) = \theta_1 X_0 + \mu_1\zeta$ and $D\varphi_{t^-(x)}(\zeta) = \theta_2 X_0 + \mu_2\zeta$, we get

$$dF(\zeta) = d\Big(\frac{a}{b}\Big)(\theta_1 X_0 + \mu_1\zeta)_{|\varphi(t^+(x))} + d\Big(\frac{a}{b}\Big)(\theta_2 X_0 + \mu_2\zeta)_{|\varphi(t^-(x))}.$$

Now it is easy to see that $d(\frac{a}{b})(X_0) = 0$. Thus, the only term that needs to be studied is $d(\frac{a}{b})(\zeta)$ and using Proposition 2.2, we have

$$d\Big(\frac{a}{b}\Big)(\zeta) = \frac{(\tilde{A} - \tilde{B})y(1-y)}{b^2}.$$

Therefore, if we show that μ_1 and μ_2 have the same fixed sign, we are done. If $\mu_1 = 0$, then ζ is transported to X_0 and since X_0 is transported along v, we have that $\zeta_x = \theta X_0$, which is impossible unless $\tilde{A} = \tilde{B}$. But in that set, $\mathcal{P} > 0$ unless $a = 0$ and hence, for $\mathcal{P} < 0$, $\mu_1 \neq 0$ and the same holds for μ_2. Now to see that they do have the same sign, it is enough to notice that any type I orbit having a point in $[\mathcal{P} < 0]$ crosses the torus defined by $\tilde{A} = \tilde{B}$ twice in a single period. Also μ_2 (or the component of the transport of ζ on ζ) is zero exactly at the crossing. Hence, μ_2 changes sign twice. Therefore, by continuity, $\mu_1 + \mu_2$ has a fixed sign in the set $[\mathcal{P} < 0]$, □

Remark 6.2. The previous construction works for $p \leq 0$, except for the circle defined by $p = 0$ and $\tilde{A} = \tilde{B}$.

Now by symmetry, we construct the map F on the orbits in $r < \frac{1}{2}$ for type I orbits by taking the intersection with s^- the symmetric of s^+.

The second step is to define F for the points in a type II orbit. Here we can define F is a similar way, i.e.,

$$F(x) = \frac{a}{b}(\varphi_{t^+(x)}(x)) + \frac{a}{b}(\varphi_{t^-(x)}(x)),$$

where $t^+(x)$ (resp. $t^-(x)$) is the first time of crossing with $s^\pm$ when flowing in the positive direction (resp. when flowing backwards) as shown in Figure 13.

Following the same procedure, we see that indeed F satisfies the properties in Proposition 6.1. Now it remains to define it for the two homoclinic orbits, and to show that F is continuous shifting from one orbit to the other. Indeed, the continuity will follow from the definition of F for the homoclinic ones since they present an intermediate configuration between type I and type II orbits.

Let us consider one homoclinic orbit. For instance, the one crossing s^+. Notice that it does cross it twice say, for instance, at two points x_1 and x_2. Then, $F(x) = \frac{a}{b}(x_1) + \frac{a}{b}(x_2)$, and by symmetry the same for the second homoclinic orbit intersecting s^-. The continuity now follows by symmetry of the orbits.

7 Periodic Orbits and Morse Index

In this section we will compute the indices of the periodic orbits of ξ and also the corresponding critical values. Recall that the index of a periodic orbit is related to the rotation of v, see [3]. So we will first study the rotation of v along ξ. Notice first that if $r_2 \neq 0, 1$, ξ is tangent to the torus $r_2 = cte$. Hence, the closed orbits corresponds to

$$\frac{\tilde{A}}{\tilde{B}} = \frac{p}{q}$$

for $p, q \in \mathbb{Z}$ with the convention that p has the same sign as $\tilde{A}$ and q has the same sign as $v\tilde{B}$. The corresponding period then is

$$\mathcal{T} = \frac{2\pi h|q|}{|\tilde{B}|} = \frac{2\pi hq}{\tilde{B}},$$

where again $h = \alpha(\zeta)$. Since X and ξ commute, we can follow the rotation of v through its projection along X, that is, b.

Along the trajectory of ξ, we have that

$$b'' + \Big(\frac{\tilde{A} - \tilde{B}}{h}\Big)^2 b = 0,$$

hence

$$b = b_0 \cos\Big(\Big|\frac{\tilde{A} - \tilde{B}}{h}\Big| t + \theta_0\Big),$$

which we can take it to be

$$b = \sqrt{r_2(1 - r_2)} \cos\Big(\Big|\frac{\tilde{A} - \tilde{B}}{h}\Big| t\Big).$$

Hence, the number of zeros of b in a period of time is given by

$$\frac{|\tilde{A} - \tilde{B}|\tau}{h\pi} = 2|p - q|.$$

Now we need to determine the direction of rotation to find out if the rotation is positive or negative. For this, we notice that $\dot{b} = \frac{(\tilde{A} - \tilde{B})}{h} a$ and from the previous computations we have

$$\dot{b} = -\sqrt{r_2(1 - r_2)} \frac{|(\tilde{A} - \tilde{B})|}{h} \sin\Big(\Big|\frac{\tilde{A} - \tilde{B}}{h}\Big| t\Big).$$

Therefore,

$$a = -\sqrt{r_2(1 - r_2)}\, \mathrm{sign}(\tilde{A} - \tilde{B}) \sin\Big(\Big|\frac{\tilde{A} - \tilde{B}}{h}\Big| t\Big).$$

Thus, the direction of rotation is determined by the sign of $(\tilde{A} - \tilde{B})$, and the index is

$$i = 2\, \mathrm{sign}(\tilde{A} - \tilde{B})|p - q|,$$

This can also read as

$$i = 2\, \mathrm{sign}(\tilde{A} - \tilde{B})|q|\Big|\frac{\tilde{A}}{\tilde{B}} - 1\Big| = 2\, \mathrm{sign}(\tilde{A} - \tilde{B}) \frac{|q|}{|\tilde{B}|} |\tilde{A} - \tilde{B}| = 2(\tilde{A} - \tilde{B}) \frac{q}{\tilde{B}} = 2(p - q).$$

The same hold for the iterated orbit, i.e.,

$$i_k = 2k(p - q).$$

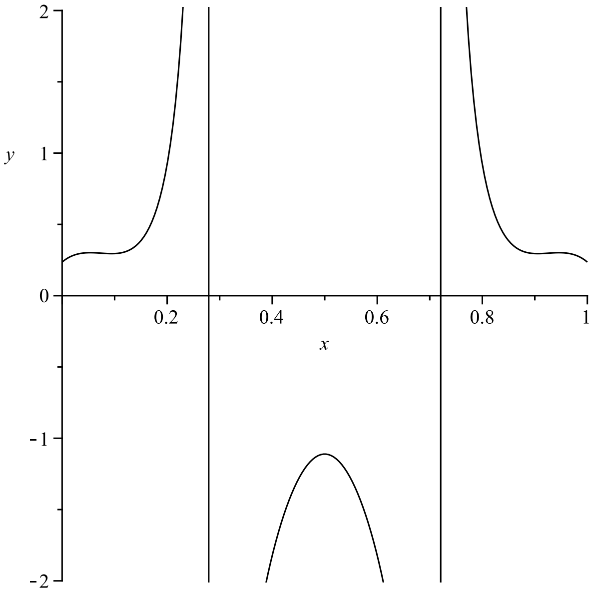

Figure 14. Critical values.

It is important to notice that the rotation of β is not determined just by the term $\tilde{A} - \tilde{B}$ as shown in the formula of Proposition 2.3. However, each periodic orbit of ξ that starts in the region $\tilde{A} - \tilde{B} < 0$ crosses the part $\mathcal{P} < 0$. That explains why the index is negative for those type of orbits. Indeed, the rotation in the part $\mathcal{P} < 0$ is greater than the rotation contained in the portion of the orbit in which $\mathcal{P} > 0$. This is because the crossing happens when $a = 0$, and if $a = 0$ and $\tilde{A} - \tilde{B} < 0$, then $\mathcal{P} < 0$.

Now if c is the corresponding critical value, we have

$$c = k \int_0^\tau \alpha(\xi)\, dt = k \frac{2\pi h p}{\tilde{A}} = 2\pi k\Big(A\tilde{B} r_1 \frac{p}{\tilde{A}} + B\tilde{A} r_2 \frac{q}{\tilde{B}}\Big) = 2\pi k(Aqr_1 + Bpr_2).$$

This can be written as

$$c = 2\pi kq(Ar_1 + Br_2) + 2\pi kBr_2(p - q).$$

Notice now that

$$i_k = 2k(p - q) = 2kq\Big(\frac{\tilde{A}}{\tilde{B}} - 1\Big),$$

and using the chosen sign convention, we have

$$i_k = 2k\frac{q}{\tilde{B}}(\tilde{A} - \tilde{B}).$$

Hence, $2kq = \frac{\tilde{B} i_k}{\tilde{A} - \tilde{B}}$, thus c reads

$$c = \pi i_k \frac{h}{\tilde{A} - \tilde{B}}.$$

This formula is not valid for $\tilde{A} - \tilde{B} = 0$ but it extends to the zero case since the index becomes zero and $c_0 = \frac{2k\pi hp}{\tilde{A}}$ and this is for only two values of r_2. In Figure 14, we can see the graph of

$$cv = \pi \frac{h}{\tilde{A} - \tilde{B}}$$

as a function of r_2, where there are two discontinuities corresponding to the zero case. The case $\tilde{A} - \tilde{B} = 0$ corresponds to closed orbits of X_0, and there is also a full circle of them since there is the action of $[\xi, v]$ that makes a full loop this time. This case is similar to the one in the torus T^3, see [6]. Another important remark is the fact that all indices are even. Hence, the circle of orbits can be split into a minimum corresponding to the strict and odd index, and a maximum of even index.

It remains to study now the case of the two circles $r_2 = 0, 1$. For this we will consider the first case, that is, $r_2 = 0$. The orbit is then of period $T_1 = \frac{2\pi h}{\tilde{B}}$ and notice that $\tilde{\tau} = (\frac{\tilde{A}-\tilde{B}}{h})^2$. Hence, if we consider the differential equation

$$\ddot{\eta} + \tilde{\tau}\eta = 0,$$

in the interval $[0, T_1]$, the number of zeros of η is at most

$$2\frac{\tilde{A} - \tilde{B}}{\tilde{B}} = 6\pi.$$

Therefore, the H_0^1 index i_0 satisfies $18 \leq i_0 \leq 19$ and the Morse index of O_0 satisfies

$$19 \leq i_{O_0} \leq 20.$$

Finally, for the iterated orbit we get

$$19k \leq i_{O_0}^k \leq 20k.$$

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