Some Dynamic and Combinatorial Properties of One Parameter Families of Unimodal Maps with Monotonicity

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Abstract: It is known that certain one parameter families of unimodal maps of the interval have a topological universality with regard to their dynamic behavior [1, 2]. As a parameter is smoothly increased, a fascinating variety of dynamic behaviors are produced. For some families the behaviors are monotonic in the parameter, while in others they are not [3]. The question is what sort of conditions on a one parameter family will ensure this monotonicity of the behavior with the parameter? The answer is unknown and will not be given here. What we do instead is to investigate certain geometric-dynamic-combinatorial consequences of assuming that the family has this monotonicity. Specifically, using tools of symbolic dynamics, state space is “course grained” with a finite alphabet. We decompose a non-invertible map into nonlinear but invertible pieces. From these invertible pieces, we form inverse maps via composition along words. Equations of motion are developed for both forward and inverse orbits (in both the variables of state space and the parameter), and an equation relating forward and inverse motions at fix-points is exhibited. Finally, we deduce a list of conditions, each of which is equivalent to monotone behavior. One of these conditions states that simple parity characteristics of words correspond to definite monotonicity. Specifically, using tools of symbolic dynamics, state space is “course grained” with a finite alphabet. We decompose a non-invertible map into nonlinear but invertible pieces. From these invertible pieces, we form inverse maps via composition along words. Equations of motion are developed for both forward and inverse orbits (in both the variables of state space and the parameter), and an equation relating forward and inverse motions at fix-points is exhibited. Finally, we deduce a list of conditions, each of which is equivalent to monotone behavior. One of these conditions states that simple parity characteristics of words correspond to definite monotonicity. Specifically, using tools of symbolic dynamics, state space is “course grained” with a finite alphabet. We decompose a non-invertible map into nonlinear but invertible pieces. From these invertible pieces, we form inverse maps via composition along words. Equations of motion are developed for both forward and inverse orbits (in both the variables of state space and the parameter), and an equation relating forward and inverse motions at fix-points is exhibited. Finally, we deduce a list of conditions, each of which is equivalent to monotone behavior. One of these conditions states that simple parity characteristics of words correspond to definite monotonicity.

Key words: One parameter family, unimodal map, kneading theory, connection equation.

1. Introduction

Let $I = [0, 1]$. We consider the collection $\xi$ one parameter families of unimodal maps $\{\mu f\}$ where $x, \mu \in I$, and $\mu f: I \to I$. $\mu f$ is at least $C^2$ in both $\mu$ and $x$. Denote the single critical point $c$, and scale the map so that $f(c) = 1$. Then $\mu f(c) = \mu$. We will need the following.

2. Brief Overview of Kneading Theory

Given $x \in I$, $O(x) = \{x, f(x), f^2(x), \ldots\}$ is called the orbit of $x$. With $O(x)$ we associate the sequence $A(x) = \{a_0, a_1, a_2, \ldots\} \in \{L, C, R\}^\mathbb{N}$ by $A(x) = \{a_0, a_1, a_2, \ldots\}$. $A(x)$ is called the itinerary of $x$ under $f$, where $a_k$ is given by

$$a_k = \begin{cases} L, & \text{for } f^k(x) < c \\ C, & \text{for } f^k(x) = c \\ R, & \text{for } f^k(x) > c. \end{cases}$$

We will be interested in studying the itinerary of $\mu f(c) = \mu$ and hence make the following definition: the itinerary of $\mu$ is called the kneading sequence of $f$, and is symbolized $K(f)$.

A kneading sequence is by convention finite precisely when it contains the letter $C$, terminating with the first $C$.

One can define a total order $<$ on the set of all kneading sequences, and more generally, on the set of all words made from the alphabet $\{L, C, R\}$, (where a word is finite if and only if it contains the letter $C$, terminating with the first $C$). This order reflects the order of the real line in the sense that $x < y$ implies $A(x) \leq A(y)$. This order is defined as follows: if $A$, $B$ are two words, and $A \neq B$, then $A < B$, $(A > B)$ according as the maximal common leading sub-word has an even (odd) number of $R$’s in it.
this order it sometimes referred to as the “parity-lexicographic” order.

A word $A$ is called maximal (or shift-maximal) provided it is greater (in the parity lexicographic order) than all of its shifts. Where, as usual, the shift operator $\sigma$ is defined by the action $\sigma(A) = a_2 a_3 a_4 \ldots$ on the word $A = a_1 a_2 a_3 \ldots$.

The significance of the kneading sequence lies in the fact that under certain conditions, for almost all $x \in I$, $O(x)$ is asymptotic to $K(f)$ [4]. Thus, under these conditions, $K(f)$ contains all information concerning the topological dynamics of $\{\mu f\}$.

It is also known that the shift-maximal sequences are just those corresponding to some $K(f)$ [5, 6].

3. Main Section

Let $\mathcal{P} = \{P: P$ is shift − maximal} and let

$\xi = \{\mu f; \mu f$ is unimodal $\land \forall P \in \mathcal{P} \forall n \geq 2 \exists \mu f\{f^n_{\mu f}(c) = c\}\}$

Recall that $f^n_{\mu f}(c) = c$ just when there is a periodic orbit of period $n$, and that $A(\mu) = P$ corresponds to this cycle.

Since $\mu f$ double covers $J = [0, \mu]$ in such a way that $\mu f([0, c]) = J = \mu f([c, 1])$, the functions $(\mu f)^{-1}, \sigma \in \{L, R\}$ have the action $(\mu f)^{-1}(J) = [0, c]$ and $(\mu f)^{-1}(J) = [c, 1]$.

Let $Q$ be a sub-pattern of $P$ obtained from $P$ by right-shifting $k − 1$ times, $1 \leq k \leq n − 1$.

Definition 1(\(\alpha\)). $x^{n-k}_Q(\mu) = (\mu f_{\sigma_{n-k}})^{-1} \circ \ldots \circ (\mu f_{\sigma_{n-1}})^{-1}(c)$.

In particular,

$x^{n-k}_Q(\mu) = (\mu f_{\sigma_{n-k}})^{-1} \circ \ldots \circ (\mu f_{\sigma_{n-1}})^{-1}(c)$.

It follows immediately that $\forall k, 1 \leq k \leq n − 1, f^n_{\mu f}(x^{n-k}_Q(\mu)) = f^n_{\mu f}(c)$, and that in particular,

$f^n_{\mu f}(x^{n-k}_Q(\mu)) = c$.

The functions $x^{n-k}_Q(\mu)$ may be constructed geometrically as follows (providing an alternate but equivalent definition):

Definition 1(\(\beta\)). Given $n \geq 2, \forall f \in \xi$ and for some $k, 1 \leq k \leq n − 1$, consider the graph of $f^{n-k}_{\mu f}$. Its extrema are in the set $\{f_\mu(c), f^{n-k}_{\mu f}(c), \ldots, f^{n-k}_{\mu f}(c)\}$ by the chain rule. When, for some $\mu, j, 1 \leq j \leq n − k, f^{j}_{\mu f}(c)$ cuts the line $y = c$, then the foot of the perpendicular dropped from the point of intersection onto the $x$-axis defines, for every $\mu$ where such an intersection exists, a function of $\mu$ denoted $x^{n-k}_Q(\mu)$.

By construction, $f^{n-k}_{\mu f}[x^{n-k}_Q(\mu)] = c$, and hence $x^{n-k}_Q(\mu)$ is an $(n − k)^{th}$-order pre-image of $c$ corresponding to some pattern $Q$ of $R$’s and $L$’s, so that

$x^{n-k}_Q(\mu) = (\mu f_{\sigma_{n-k}})^{-1} \circ \ldots \circ (\mu f_{\sigma_{n-1}})^{-1}(c)$,

where $Q = \sigma_1 \sigma_2 \ldots \sigma_{n-k}$. In particular, for $P = \sigma_1 \sigma_2 \ldots \sigma_{n-1}$,

$x^{n-k}_P(\mu) = (\mu f_{\sigma_1})^{-1} \circ \ldots \circ (\mu f_{\sigma_{n-1}})^{-1}(c)$.

Call the set of these $x^{n-k}_Q(\mu)$, where $Q$ corresponds to a right shift of some $P \in \mathcal{P}, \mathcal{A}$. Then the set $\mathcal{A}$ and the collection of the functions defined in definition $1(\alpha)$ are identical.

Remark 1. Denote the value of $\mu$ such that $f^{j}_{\mu f}(c) = c$ by $\mu^*_Q$, or, when $Q$ is understood, simply by $\mu^*$. By the implicit function theorem, $\mu^*$ is the parameter value at which $x^{n-k}_Q(\mu)$ comes into existence.

Definition 2. Call the $s$-order level function with the property $f^{n-k}_{\mu f}[x^s(\mu)] = f^{j}_{\mu f}(c)$ the support of $f^{j}_{\mu f}(c), 1 \leq k \leq n − k$. (we have seen above that $s = n + k − j$)

Remark 2. Note that the left-end of domain of $x^{n-k}_Q$ is open: $x^{n-k}_Q(\mu^*_Q) = \lim_{\mu \rightarrow \mu^*_Q} x^{n-k}_Q(\mu)$.

Remark 3. Since between any two adjacent extrema (where these extrema are on opposite sides of the line $y = c$) there is a unique level function, and because the $x$-order of the critical numbers lying under extrema is fixed, the $x$-ordering of the various level functions is also fixed. Therefore, their trajectories never intersect.

I will argue in the sequel that, given certain conditions on $f \in \xi$, uniqueness of values of the
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Fig. 1  Illustration of Level Functions for \( \mu f(x) \)

scaling parameter corresponding to any \( f_\mu^2(c), \ldots, f_\mu^{n-k} P \in \mathcal{P} \) is characterized by a set of dynamic and combinatorial conditions. Namely, that for all \( f \in \xi \) and \( n \geq 2 \), and each \( P \in \mathcal{P} \) the following are equivalent:

(i) There exists a unique \( \mu \) such that \( x_{\mu}^{n-1}(\mu) = \mu \)

(ii) \( x_{\mu}^{n-1}(\mu) = \mu \Rightarrow \frac{d}{d\mu} x_{\mu}^{n-1}(\mu) < 1 \)

(iii) \( f_{\mu}^{n}(c) = c \)

\[
\begin{align*}
\frac{d}{d\mu} f_{\mu}^{n}(c) > 0 & \iff \tau(P) \equiv 0 \mod 2 \\
\frac{d}{d\mu} f_{\mu}^{n}(c) < 0 & \iff \tau(P) \equiv 1 \mod 2
\end{align*}
\]

Where \( \tau \) counts the number of R’s in \( P \).

(iv) \( f_{\mu}^{n}(c) = c \Rightarrow \left( \frac{d^2}{dx^2} f_{\mu}^{n}(x) \right)_{x=c} \frac{d}{d\mu} f_{\mu}^{n}(c) < 0 \)

Note that the curvature \( \kappa \) of any function \( f \) at \( x \) is given by

\[
\kappa = \frac{d^2}{dx^2} f(x) \frac{1}{\left[1 + \left( \frac{d}{dx} f(x) \right)^2 \right]^{3/2}}
\]

But at \( x = c, \frac{d}{dx} f_\mu^{n}(x) = 0 \). Consequently, for smooth unimodal maps,

\[
\kappa = \frac{d^2}{dx^2} f_\mu^{n}(x) \Big|_{x=c}.
\]

Therefore, condition (iv) says that at super-stable points, that is, where \( f_\mu^{n}(c) = c \), \( f_\mu^{n}(c) \) has a non-zero \( \mu \)-motion opposite to the signed curvature of \( f_\mu^{n}(x) \) at \( x = c \).

Proposition 1 For all \( f \in \xi \) and for each \( k, 1 \leq k \leq n - 1 \),

\( f_\mu^{n}(c) = c \iff x^{n-k}(\mu) = f_\mu^{k}(c) \). In particular,

\( f_\mu^{n}(c) = c \iff x^{n-1}(\mu) = \mu \).

Suppose \( f_\mu^{k}(c) = c \). Then \( f_\mu^{k}(c) = x \in [0,1] \). But then \( f_\mu^{n-k}[f_\mu^{k}(c)] = f_\mu^{n}(c) = c \), so that \( x \) is an \( (n - k)^{th} \) pre-image of \( c \). That is, \( x = x^{n-k}(\mu) \).

What follows is a sequence of technical lemmas.

Lemma 1. For all \( f \in \xi \) and \( n \geq 2 \), and for each \( P \in \mathcal{P} \), there exists a unique \( \mu \) such that \( x_{\mu}^{n-1}(\mu) = \mu \Rightarrow \text{dom} x_{\mu}^{n-1}(\mu) \) is connected.

Fig. 2  Illustration of level functions for \( f_\mu^{2}(x) \).
Proof. Suppose that dom $x_p^{n-1}(\mu)$ is not connected. Then by definition 1($\beta$),
\[ \exists \mu_0, \mu_1, j > 1 \left[ f_{\mu_0}^j(c) = c = f_{\mu_1}^j(c) \right]. \]
But using proposition 1 (with $j = n$) we have
\[ x_p^{n-1}(\mu_0) = f_{\mu_0}^n(c) = \mu_0 \text{ and } x_p^{n-1}(\mu_1) = f_{\mu_1}^n(c) = \mu_1, \text{ where } P' \in \mathcal{P} \text{ of order } j, \text{ which contradicts the hypothesis.} \]

Corollary. Given the hypothesis of proposition 2, dom $x_p^{n-1}(\mu) = (\mu_p, 1)$. This follows from definition 1($\beta$) and remarks 1 and 2 following it.

Lemma 2. For all $f \in \xi$ and $n \geq 2$, and for each $P \in \mathcal{P}$ and for $\mu > \mu_R$
\[ x_p^{n-1}(\mu) > \mu \Leftrightarrow x_p^{n-2}(\mu) < f_{\mu}^2(c). \]

Proof. $x_p^{n-1}(\mu) = \mu \Rightarrow \mu f[x_p^{n-1}(\mu)] < \mu f_R(\mu)$ since $\mu f_R$ reverses orientation. But $\mu f_R[x_p^{n-1}(\mu)] = x_p^{n-2}(\mu)$ and $\mu f_R(\mu) = f_{\mu}^2(c)$, so that $x_p^{n-2}(\mu) < f_{\mu}^2(c)$. On the other hand, $f_{\mu}^{-1}[x_p^{n-2}(\mu)] = x_p^{n-1}(\mu)$ and $(\mu f_R)^{-1}[f_{\mu}^2(c)] = \mu f(c)$. So again, as $\mu (f_R)^{-1}$ reverses orientation, we have $x_p^{n-1}(\mu) > \mu$. (observe that $(\mu f_R)^{-1}$, $(\mu f_R)^{-1}$ are both defined and equal at $c$, so $(f_R)^{-1}$ is defined at $c$.)

Lemma 3. For all $f \in \xi$ and for all $\mu > \mu_R$
\[ f'(\mu) < -\frac{\mu}{2} f''(\mu) \Leftrightarrow \frac{d^2}{d\mu^2} f_{\mu}^2(c) < 0 \]

Proof. $\frac{d}{d\mu} f_{\mu}^2(c) = f(\mu) + \mu f'(\mu)$ so that
\[ \frac{d^2}{d\mu^2} f_{\mu}^2(c) = 2f'(\mu) + \mu f''(\mu). \]

Note concavity of $f$ is a sufficient (but stronger) hypothesis.

Lemma 4. For all $f \in \xi$, $n \geq 2$ and $k$, $1 \leq k \leq n - 1$, and for $\mu > \mu_R$
\[ \left[ f_{\mu}^n(c) = c \land \frac{d}{d\mu} f_{\mu}^n(c) = 0 \right] \Leftrightarrow \left[ x_p^{n-1}(\mu) = \mu \land \frac{d}{d\mu} x_p^{n-k}(\mu) \right] = \frac{d}{d\mu} f_{\mu}^k(c). \]

Proof. By proposition 1 $f_{\mu}^k(c) = c \Leftrightarrow x^{n-k}(\mu) = f_{\mu}^k(c)$. Suppose, therefore, that $\frac{d}{d\mu} f_{\mu}^n(c) = 0, f_{\mu}^n(c) = c$. Then
\[ \mu f^n[x^1(\mu)] = c \land \forall \mu \in N \]
\[ \Rightarrow f[x^1(\mu)] + \mu f'[x^1(\mu)] \frac{d}{d\mu} x^1(\mu) = 0 \]
so that
\[ \frac{d}{d\mu} x^1(\mu) = -\frac{f[x^1(\mu)]}{\mu f'[x^1(\mu)]}, \]
\[ \mu f[x^2(\mu)] = x^1(\mu) \land \forall \mu \in N \]
\[ \Rightarrow f[x^2(\mu)] + \mu f'[x^2(\mu)] \frac{d}{d\mu} x^2(\mu) = x^1(\mu), \]
so that
\[ \frac{d}{d\mu} x^2(\mu) = \frac{\mu f[x^1(\mu)] - f[x^2(\mu)]}{\mu f'[x^2(\mu)]} = \frac{-f[x^1(\mu)]}{\mu f'[x^1(\mu)]} \]
and by induction,
\[ \frac{d}{d\mu} x^k(\mu) = \frac{-f[x^k(\mu)]}{\mu f'[x^k(\mu)]} = \cdots = \frac{-f[x^{k-1}(\mu)]}{\mu f'[x^{k-1}(\mu)]} \]

Now
\[ \frac{d}{d\mu} f_{\mu}^2(c) = f(\mu) + \mu f'(\mu) \]
\[ \frac{d}{d\mu} f_{\mu}^3(c) = f[f_{\mu}^2(c)] + \mu f'[f_{\mu}^2(c)] \frac{d}{d\mu} f_{\mu}^2(c) \]
\[ \vdots \]
\[ \frac{d}{d\mu} f_{\mu}^n(c) = f[f_{\mu}^{n-1}(c)] + \mu f'[f_{\mu}^{n-1}(c)] \frac{d}{d\mu} f_{\mu}^{n-1}(c) \]
\[ \Rightarrow f[x^1(\mu)] + \mu f'[x^1(\mu)] \frac{d}{d\mu} x^1(\mu) = \frac{d}{d\mu} f_{\mu}^n(c) = 0 \]

conversely:

Suppose $x_p^{n-1}(\mu) = \mu$ and $\frac{d}{d\mu} x_p^{n-1}(\mu) = \frac{d}{d\mu} f_{\mu}^k(c)$ then by proposition 1,
\[ \frac{d}{d\mu} f_{\mu}^n(c) = f[x^1(\mu)] + \mu f'[x^1(\mu)] \frac{d}{d\mu} f_{\mu}^{n-1}(c) \]
\[ = f[x^1(\mu)] + \mu f'[x^1(\mu)] \frac{d}{d\mu} x^1(\mu) \text{ by hypothesis.} \]
But \( \frac{d}{d \mu} x^1(\mu) = \frac{-f'[x^1(\mu)]}{\mu f'[x^1(\mu)]} \)

by substitution, \( \frac{d}{d \mu} f^1_n(c) = 0. \)

continuing:

\[
\frac{d}{d \mu} x^1(\mu) = \frac{d}{d \mu} f^1_n(c) = f[f^1_n(c)] + \mu f'[x^1(\mu)] \frac{d}{d \mu} f^{n-2}(c) = f[x^2(\mu)] + \mu f'[x^2(\mu)] \frac{d}{d \mu} f^{n-2}(c) \text{ so that}
\]

\[
\frac{d}{d \mu} f^{n-2}(c) = \frac{\frac{d}{d \mu} x^1(\mu) - f[x^2(\mu)]}{\mu f'[x^2(\mu)]} = \frac{-f[x^1(\mu)]}{\mu f'[x^2(\mu)]} - \frac{f[x^1(\mu)]}{\mu f'[x^2(\mu)]}.
\]

The result follows by induction.

**Lemma 5.** For all \( f \in \mathcal{F} \) and \( n \geq 2 \), and each \( P \in \mathcal{P} \), there exists a unique \( \mu \) such that

\[ x^{n-1}_P(\mu) = \mu \Rightarrow \frac{d}{d \mu} x^{n-1}_P(\mu) \leq 1. \]

**Proof.** Lemma 1 and the uniqueness assumption.

**Lemma 6.** If for each \( x > c, f'(x) < -\frac{\mu}{2} \), then

\[
\frac{d}{d \mu} x^{n-2}(\mu) = \frac{\frac{d^2}{d \mu^2} x^{n-2}(\mu) - 2f'(\mu) - \mu f(\mu)}{\mu f'(\mu)} < 0.
\]

since \( x^{n-1}(\mu) = \mu \) and \( \frac{d}{d \mu} x^{n-1}(\mu) = 1. \) But then

\[
\frac{d^2}{d \mu^2} x^{n-2}(\mu) = 0 \text{ implies that}
\]

\[
\frac{d^2}{d \mu^2} x^{n-2}(\mu) - 2f'(\mu) - \mu f(\mu) = 0.
\]

Therefore, \( \frac{d^2}{d \mu^2} x^{n-2}(\mu) = -2f'(\mu) - \mu f'(\mu) < 0 \),

(by hypothesis).

By Lemmas 2, 3, 4 and the uniqueness assumption, we must have that \( \frac{d^2}{d \mu^2} x^{n-2}(\mu) = 0. \) This is because these propositions together imply that \( x^{n-2}(\mu) \) agrees with \( f_\mu^2(c) \) both in position and derivative at the value of \( \mu \) where \( x^{n-1}_P(\mu) = \mu. \)

Therefore, \( x^{n-2}(\mu) \) must have "cubic" behavior on the tangent line to the trajectory of \( f_\mu^2(c) \) at the value of \( \mu \) where \( x^{n-1}_P(\mu) = \mu. \) That is,

\[
\frac{d}{d \mu} x^{n-2}(\mu) = \frac{d}{d \mu} f^2(c) \quad \text{and} \quad \frac{d^2}{d \mu^2} x^{n-2}(\mu) = 0.
\]

Therefore, \( \frac{d}{d \mu} x^{n-1}(\mu) \neq 1 \Rightarrow \frac{d}{d \mu} x^{n-1}(\mu) < 1. \)

**Note** that \( \forall f \in \mathcal{F}, \forall x > c, f'(x) < -\frac{\mu}{2} \), could be replaced by the stronger statement

\( \forall f \in \mathcal{F}, f''(x) < 0. \)

Since in proposition 4, the case \( n = 1, \mu = \mu_R \) is not handled, but proposition 4 is used essentially in proposition 8, we prove here that \( x^1_R(\mu) = \mu \Rightarrow \frac{d}{d \mu} x^1_R(\mu) < 1: \)

\[
\forall \mu, \mu f[x^1_R(\mu)] = c, \text{ so that when } x^1_R(\mu) = \mu
\]
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\[
\frac{d}{d\mu} x^1(\mu) = -f[x^1(\mu)] = -f(\mu)
\]

Now if \( \forall \mu > \mu_R, f'(\mu) < -1 \), we have

\[
\frac{d}{d\mu} x^{n-1}(\mu) < 1.
\]

**Note:** \( \mu f'(\mu) < -1 \Leftrightarrow f'(\mu) < -\frac{1}{\mu} \) and \( f'(\mu) < -\frac{\mu}{2} F'(x) \) by the hypothesis of proposition 8.

\[
-\frac{\mu}{2} f''(x) \Rightarrow f''(\mu) < \frac{1}{\mu} \Leftrightarrow f''(\mu) < -\frac{1}{\mu}.
\]

Therefore, \( -\frac{\mu}{2} f''(\mu) > -\frac{1}{\mu} \Rightarrow \mu f'(\mu) < -1 \). This certainly holds if \( \forall x, f''(x) < 0 \).

**Lemma 7.** For all \( f \in \xi \) and \( n \geq 2 \), and each \( P \in \mathcal{P} \), there exists a unique \( \mu \) such that

\[
x^{n-1}_p(\mu) = \mu \Rightarrow \frac{d}{d\mu} f^n_p(\mu) \neq 0
\]

**Proof.** An easy induction yields, \( \forall n \geq 2 \forall \mu > \mu_R \),

\[
\frac{d}{d\mu} f^n_p(\mu) = \frac{d}{d\mu} \left( f^{n-1}_p(\mu) \right) + \mu f'(\mu) f^{n-1}_p(\mu)
\]

and, \( \forall k, 1 \leq k \leq n-1 \)

\[
\frac{d}{d\mu} f^k_p(\mu) = \frac{f[k^1(\mu)]}{\mu^k} \frac{f[k^2(\mu)]}{\mu^k} \cdots \frac{f[k^1(\mu)]}{\mu^k}
\]

It follows from proposition 1 and (*) that

\[
\frac{d}{d\mu} f^n_p(\mu) = \frac{f[k^1(\mu)]}{\mu^k} \frac{f[k^2(\mu)]}{\mu^k} \cdots \frac{f[k^1(\mu)]}{\mu^k}
\]

when \( x^{n-1}(\mu) = \mu \).

Dividing both sides by the last term on the RHS, we have

\[
\frac{d}{d\mu} f^n_p(\mu) = \frac{f[k^1(\mu)]}{\mu^k} \frac{f[k^2(\mu)]}{\mu^k} \cdots \frac{f[k^1(\mu)]}{\mu^k}
\]

This last form is called the connection equation, as it relates the \( \mu \) and \( x \) motion of forward iterates with “inverse iterates” at parameter values corresponding to super-stable points.

But \( \frac{d}{d\mu} x^{n-1}(\mu) \neq 1 \) whenever \( x^{n-1}(\mu) = \mu \) by

**Lemma 6.** Therefore, \( \frac{d}{d\mu} f^n_p(\mu) \neq 0 \) whenever

\[
f^n_p(\mu) = c \text{ by proposition 1.}
\]

Now suppose that \( P \in \mathcal{P} \). Let \( \tau: \mathcal{P} \to \mathbb{N} \) count the number of \( R' s \) in \( P \).

**Proposition 2.** For all \( f \in \xi \) and \( n \geq 2 \), and each \( P \in \mathcal{P} \),
The connection equation, proposition 8, and the definition of a unimodal map.

**Corollary.** For all $f \in \xi$ and $n \geq 2$, and each $P \in \mathcal{P}$, there exits a unique $\mu$ such that $x_{\mu}^{-1}(\mu) = \mu$

$$f_{\mu_P}^{n}(c) = c \Rightarrow \begin{cases} \frac{d}{d\mu} f_{\mu}^{n}(c) > 0 \iff \tau(P) \equiv 0 \text{ mod } 2 \\ \frac{d}{d\mu} f_{\mu}^{n}(c) < 0 \iff \tau(P) \equiv 1 \text{ mod } 2 \end{cases}$$

**Proof.** By remark 4 there exists a $\mu$-nbh $N$ over which $\tau(P)$ is invariant. Therefore, $\forall \nu \in N, \text{sgn} \frac{d}{d\mu} f_{\nu}^{n}(c)$ is invariant by hypothesis. Consequently, $\forall \nu \in N, \frac{d}{d\mu} f_{\nu}^{n}(c) \neq 0$ and hence $\frac{d}{d\mu} f_{\nu}^{n}u(c)$ cannot reverse its motion (which would be necessary in order that there be another value of $\mu$ (corresponding to $P$) such that $f_{\mu}^{n}(c) = c$). Therefore, $\mu$ such that $f_{\mu}^{n}(c) = c$ is unique.

**Proposition 4.** For all $f \in \xi$ and $n \geq 2$, and each $P \in \mathcal{P}$,

$$f_{\mu_P}^{n}(c) = c \Rightarrow \begin{cases} \frac{d}{d\mu} f_{\mu}^{n}(c) > 0 \iff \tau(P) \equiv 0 \text{ mod } 2 \\ \frac{d}{d\mu} f_{\mu}^{n}(c) < 0 \iff \tau(P) \equiv 1 \text{ mod } 2 \end{cases}$$

**Proof.** A straightforward induction yields the equation

$$\frac{d^2}{dx^2} f_{\mu}^{n}(c)|_{x = c} = -\mu^n \frac{d^2}{dx^2} f(x)|_{x = f_{\mu}^{n-k}(c)} \quad \text{for } n \geq 2, \quad \text{by (2), the definition of } \rho$$

Define $\rho: \mathbb{N} \rightarrow \{-1,1\}$ by $\rho[\tau(P)] = \pm 1$ according as $\tau(P) \equiv 0,1 \text{ mod } 2$. Then

$$\tau(P) = -\text{sgn} \frac{d^2}{dx^2} f_{\mu}^{n}(c)(2)$$

since $\rho[\tau(P)] = \text{sgn} \prod_{k=1}^{n-1} \frac{d}{dx} f_{\mu}^{n}(c)$ and by hypothesis, $\frac{d^2}{dx^2} f_{\mu}^{n}(c)|_{x = c} < 0$.

Now suppose that $f_{\mu}^{n}(c) = c$ and $\frac{d^2}{dx^2} f_{\mu}^{n}(c)|_{x = c} \frac{d}{d\mu} f_{\mu}^{n}(c) < 0$. Then, in particular, $\frac{d^2}{dx^2} f_{\mu}^{n}(c)|_{x = c} \neq 0$ so that $\tau(P) \equiv 0 \text{ mod } 2$ implies $\frac{d}{d\mu} f_{\mu}^{n}(c) > 0$ by (2), the definition of $\rho$, and the hypothesis. $\tau(P) \equiv 1 \text{ mod } 2$ $\Rightarrow \frac{d}{d\mu} f_{\mu}^{n}(c) < 0$ for the same reason.

Suppose that for all $f \in \xi$ and $n \geq 2$, and each $P \in \mathcal{P}$,

$$f_{\mu_P}^{n}(c) = c \Rightarrow \begin{cases} \frac{d}{d\mu} f_{\mu}^{n}(c) > 0 \iff \tau(P) \equiv 0 \text{ mod } 2 \\ \frac{d}{d\mu} f_{\mu}^{n}(c) < 0 \iff \tau(P) \equiv 1 \text{ mod } 2 \end{cases}$$

But $\tau(P) \equiv 0 \text{ mod } 2$ implies $\frac{d}{d\mu} f_{\mu}^{n}(c) > 0$ by hypothesis and

$$\rho[\tau(P)] = -\text{sgn} \frac{d^2}{dx^2} f_{\mu}^{n}(c) \Rightarrow \text{sgn} \frac{d^2}{dx^2} f_{\mu}^{n}(c) > 0$$

so that $\frac{d^2}{dx^2} f_{\mu}^{n}(c) \frac{d}{d\mu} f_{\mu}^{n}(c) < 0$. Similarly, $\tau(P) \equiv 0 \text{ mod } 2$
$1 \mod 2 \Rightarrow \frac{d}{d\mu} f^n_\mu(c) < 0$ by hypothesis and

$$\rho[\tau(P)] = -\text{sgn} \frac{d^2}{dx^2} f^n_\mu(c) \Rightarrow \text{sgn} \frac{d^2}{dx^2} f^n_\mu(c) > 0.$$ so that $\frac{d^2}{dx^2} f^n_\mu(c) \frac{d}{d\mu} f^n_\mu(c) < 0$.

Thus, this cycle of propositions proves the claimed equivalence of relations $(i)$ through $(iv)$ --

Namely, $(i)$ implies $(iii)$ by the corollary to proposition 2; $(iii)$ implies $(i)$ by propositions 1 and 3; $(ii)$ is equivalent to $(iii)$ by proposition 2 and the connection equation; and $(iii)$ is equivalent to $(iv)$ by proposition 4.

References